1. (24 points) There are 999,999 natural numbers less than one million. We write any of them as a six digit number, including leading zeros. (For example, 001124 is how we write the number 1124).

(a) How many of these numbers have all different digits?

all diff digits means no repetitions. Order matters!

10 ~~9~~ 9 8 7 6 5 By Multiplication Principle,

$\boxed{\frac{10!}{4!}}$

+8

(b) How many of these numbers have exactly four distinct digits? (For example, 922433 is valid, but 922435 is not valid and 922444 is not valid).

~~1) choose 4 distinct digits.~~

10 9 8 7 ~~4~~ 6 ~~4~~ 6

pick 4 distinct.

Remaining 2 can repeat.

~~or 10 · 9 ...~~

$\boxed{\frac{10!}{5!} \cdot 6}$

+2

(c) How many of these numbers have digits that sum to 18?

$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 18$

Restrict $0 \le x_i \le 9$ for $1 \le i \le 6$.

$\boxed{C(23,18) - 6C(13,8) \text{ numbers}}$

+8

W/o restrictions, $C(23,18)$ ways.

Subtract the complements, e.g. $x_1 \ge 10$, $x_2 \ge 10$ ... $x_6 \ge 10$ separately.

$(x_1 + 10) + x_2 + \cdots x_6 = 18$

$x_1 + x_2 + \cdots x_6 = 8$

↳ $C(13,8)$ ways

2. (16 points) Solve the recurrence relation $A_n = 3A_{n-1} + 4A_{n-2}$, where $A_0 = 3$, $A_1 = 7$.

Char. poly:

$t^2 - 3t - 4 = 0$

$(t-4)(t+1) = 0$

$t = 4, -1$ Distinct roots.

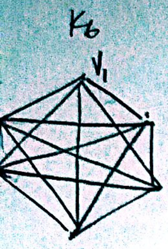
$A_n = b(4)^n + d(-1)^n$

Initial conditions:

$A_0 = 3 = b + d$

3. (22 points) Recall that a k-cycle is a cycle that includes k edges. In this problem, you will prove Ramsey's theorem, which states that if $n \geq 6$ and we color each edge of K_n either blue or red, then there must exist either a set of three blue edges that form a 3-cycle, or a set of three red edges that form a 3-cycle.
To this end, let $n \geq 6$ be arbitrary, and suppose every edge in K_n is colored either blue or red. Let v_1 be a vertex in K_n.

Prove that at least three of the edges incident to v_1 are the same color.

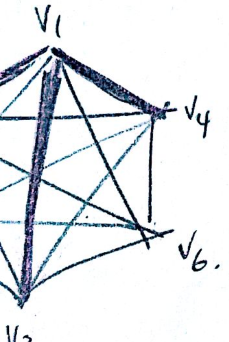


Proof:
By definition, K_n is the complete graph w/ n vertices, meaning all vertices are connected to all others. For K_n, this means each vertex has degree $(n-1)$, or $(n-1)$ edges incident on it. We prove that for $n \geq 6$, ~~...~~ at least 3 edges incident to v_1 (or any vertex) are the same ~~color by induction~~ color by the pigeonhole principle. Suppose we have

2

2 ~~pigeon~~ pigeonholes, one blue and one red. ~~#~~ And suppose there are $n=6$ vertices, in K_6. This gives $(6-1)=5$ edges coming out of v_1. Let the edges be pigeonholes. By the third form of the PHP, at least $\lceil \frac{5}{2} \rceil = 3$ pigeons will be in the same hole (either red or blue). By a similar argument, any $n \geq 6$, will yield $\lceil \frac{n}{2} \rceil \geq 3$. Therefore, at least 3 edges incident on v_1 are blue or red.

(b) In the previous part, you proved that at least three of the edges incident to v_1 are the same color. Without loss of generality, you may assume that color is blue. Suppose that $\{v_1, v_2\}$, $\{v_1, v_3\}$, and $\{v_1, v_4\}$ are blue edges. Prove that between these four vertices, there must exist either a blue 3-cycle or a red 3-cycle. (can't include all 4) .



Proof: Suppose we begin at vertex v_1. We take the blue edge to vertex v_2. ~~By~~ Because K_n is connected to all other vertices, ~~we take~~ and at least 3 edges are the same color (blue ~~...~~). There are several cases:

4 → if you try to consider v_5 and v_6, you'll get an... by cases.

① ~~3~~ remaining edges from v_2 are red. 0 blue.
No blue-3-cycle exists. Then start at v_2 and project to v_3 and v_4.
...essarily →
Either v_3 and v_4 have a red edge b/t them, or they have a blue edge. Either way, a blue 3-cycle exists b/t (v_1, v_3, v_4)
8
red $(v_2, v_3, ...$

② ~~3~~ remaining edges are red; 1 is blue.
Similar argument as ①: Take v_2 to v_3 to v_4. Either a cycle opens up, the red between (v_2, v_3, v_4) or a blue cycle between (v_1, v_3, v_4). ok

③ ~~3~~ ~~4~~ 2 or more are blue. Then one of these edges is also
why? could both be incident on v_5/v_6. This doesn't cover everyth...
incident on v_3 or v_4 by PHP, 3rd form. Then knowing a blue edge connects v_1 to v_2, v_2 to v_3, and v_3 to v_1, we have found a blue 3-cycle

4. **(18 points) Prove the combinatorial identity**

$$\sum_{i=0}^{k} C(m+k-i-1, k-i) \cdot C(n+i-1, i) = C(m+n+k-1, k)$$

using a combinatorial argument. No more than half credit will be awarded to an algebraic proof. (Hint: Use Pirates and Gold.)

Let there be 2 groups of pirates, one of size m, and the other of size n, both fighting over k pieces of gold. We can calculate the number of ways to distribute the gold between them in the 2 following ways.

① RHS:

There are m+n total pirates and k pcs. of gold. By definition, we can choose $C((m+n)+k-1, k)$ ways to divy the gold.

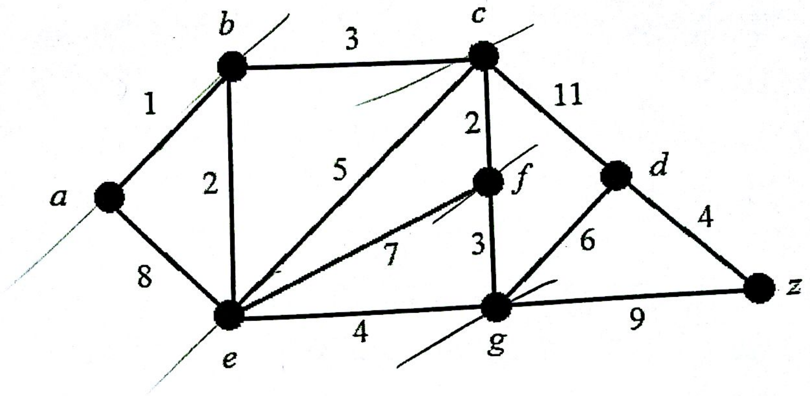
② LHS:

Partition the ways to pass out gold by the number of pieces, i, that the group of n pirates receives. i can range from 0 to as high as k. First, distribute i pieces of gold to the n pirates. Then distribute the remaining (k-i) pieces to the other group of m pirates. This gives us $C(n+i-1, i)$ and $C(m+k-i-1, k-i)$, respectively. Because the pieces of gold, i, breaks up the set of ways into disjoint we can sum them up from 0 to k. This gives $\sum_{i=0}^{k} C(m+k-i$ $\cdot$ (

k

total

5. (20 points) Run Dijkstra's algorithm on the following graph to find the shortest path from a to z. Recall that at each stage of Dijkstra's algorithm, one vertex is chosen and given a permanent label which represents the length of the shortest path from a to that vertex. Write down the list of vertices in the order in which they are given permanent labels. Additionally, find the length of a shortest path from a to z.



i. 1 $L(a) = 0$

$T = \{a, b, \ldots, z\}$ $\ell(b), \ell(c), \ldots, \ell(z) = \infty$

$\ell(b) = 1, \ \ell(e) = 8$

ii. 2 $L(b) = 1$

$\ell(e) = \min(8, 3) = 3$ $\ell(c) = 4$

iii. 3 $L(e) = 3$

$\ell(c) = 4, \ \ell(f) = 10, \ \ell(g) = 7$

iv. 4 $L(c) = 4$

$\ell(f) = \min(10, 6) = 6, \ \ell(d) = 15$

v. 5 $L(f) = 6$

$\ell(g) = \min(7, 9) = 7$

vi. 6 $L(g) = 7$

$\ell(d) = \min(15, 13) = 13$ $\ell(z) = 16$

Order of vertices:

$\boxed{a, b, e, c, f, g, d, z}$

Length of shortest path

$\boxed{L(z) = 16}$