Math 61, Lec 1
Winter 2016
Exam 2
2/22/16
Time Limit: 50 Minutes

Name

Nam

Discussion

This exam contains 6 pages (including this cover page) an
are missing.

You may *not* use books, notes, or any calculator on this exam.

Unless otherwise stated in the problem, you may leave all answers in terms of $\binom{n}{k}$, $P(n,k)$, $k!$, or any sum, difference, product, or quotient of such symbols.

Partial credit will only be awarded to answers for which an explanation and/or work is shown.

Please attempt to organize your work in a reasonably neat and coherent way, in the space provided. If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score	
1	24	9	15
2	16	16	
3	22	22	
4	18	18	
5	20	20	
Total:	100	85	

1. (24 points) There are 999,999 natural numbers less than one million. We write any of them as a six digit number, including leading zeros. (For example, 001124 is how we write the number 1124).

(a) How many of these numbers have all different digits?

$6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = \boxed{6! \text{ choices}}$

↑ 1st ↑ 2nd ↑ 3rd ↑ 4th ↑ 5th ↑ 6th

+1

(b) How many of these numbers have exactly four distinct digits? (For example, 922433 is valid, but 922435 is not valid and 922444 is not valid).

$\underbrace{6 \cdot 6}_{\text{whatever digits}} \; \underbrace{6 \cdot 5 \cdot 4 \cdot 3}_{\text{4 distinct digits}}$

$= 36 \cdot \frac{6!}{2!}$

+ 2

(c) How many of these numbers have digits that sum to 18?

$x_6 + x_5 + x_4 + x_3 + x_2 + x_1 = 18$

$\to C(18+6-1, 6-1) = \boxed{C(23, 5)}$

+6

2. (16 points) Solve the recurrence relation $A_n = 3A_{n-1} + 4A_{n-2}$, where $A_0 = 3$, $A_1 = 7$.

$t^2 = 3t + 4$

$\to t^2 - 3t - 4 = 0 \Rightarrow (t-4)(t+1)$

$, r = 4, -1$

① $3 = C_1 + C_2$

② $7 = 4C_1 + {}^{-}C_2$

①+② $\Rightarrow 10 = 5C_1$

$C_1 = 2, \; C_2 = 1$

$\to \boxed{A_n = 2(4)^n + 1(-1)^n}$

3. (22 points) Recall that a k-cycle is a cycle that includes k edges. In this problem, you will prove Ramsey's theorem, which states that if $n \geq 6$ and we color each edge of K_n either blue or red, then there must exist either a set of three blue edges that form a 3-cycle, or a set of three red edges that form a 3-cycle.
To this end, let $n \geq 6$ be arbitrary, and suppose every edge in K_n is colored either blue or red. Let v_1 be a vertex in K_n.

Prove that at least three of the edges incident to v_1 are the same color.

bad word

Assumption: For all $n \geq 6$, K_n has all vertices with at least 3 edges same color

Negation: For all vertices, there are at most 2 edges with same color

Pf. Let K_n be complete graph with $n \geq 6$. Let $V_1 \in K_n$ be arbitrary. Since V_1 has at most 2 edges w/ same color, then V_1 has 4 edges at most, since there are only 2 colors. However, V_1 has $n-1$ edges, and since $n \geq 6$, $n-1 > 4$. So, V_1 cannot have at most 2 colored edges of each color.

What is the major result from class that you prove but manage to avoid using here?

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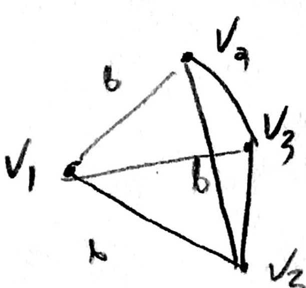
(b) In the previous part, you proved that at least three of the edges incident to v_1 are the same color. Without loss of generality, you may assume that color is blue. Suppose that $\{v_1, v_2\}$, $\{v_1, v_3\}$, and $\{v_1, v_4\}$ are blue edges. Prove that between these four vertices, there must exist either a blue 3-cycle or a red 3-cycle.



Given: $V_1 \in G$ has blue edges to V_2, V_3, and V_4.

bad word

Assumption: G has blue or red 3-cycle

Pf: Let G be a connected graph K_4 with the above specification. Because V_1 has blue edges to $V_2, V_3 \& V_4$, any blue edge between two of the vertices would create a blue cycle in the graph. Therefore, any combination of edges between $V_2, V_3, \& V_4$ containing 1 blue edge has a colored 3-cycle. In addition, if there are no blue edges between $V_2, V_3, \& V_4$, then, because there are only 2 possible colors, a red 3-cycle is created. Because a cycle is always present no matter the color combination of remaining edges, this is always true.

Maybe you mean "Assertion" instead of "Assumption"?

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4. (18 points) Prove the combinatorial identity

$$\sum_{i=0}^{k} C(m+k-i-1,k-i)\cdot C(n+i-1,i) = C(m+n+k-1,k)$$

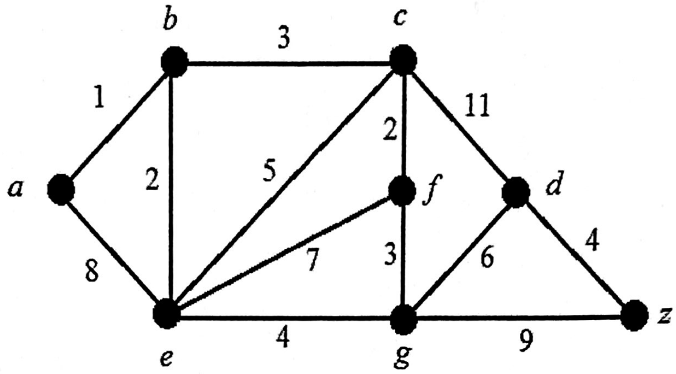
using a combinatorial argument. No more than half credit will be awarded to an algebraic proof. (Hint: Use Pirates and Gold.)

Let m and n be separate quantities of identical scoops of ice cream, and we want to distribute them among k sundaes. Using basic Pirates and Gold, with there being k "pirates" (sundaes) and m+n "pieces of Gold" (scoops of icecream), we get that there are $C(m+n+k-1, k)$ ways to distribute these scoops of ice cream.

However, we can also consider each partition of splitting the n scoops into a first group of i cups, and then splitting the remainder of the scoops in the remaining k-i cups. For every $0 \le i \le k$, this would be $C(n+i-1, i)$ for the distribution among the first i cups (using P&G again), and $C(m+k-i-1, k-i)$ to distribute the m-scoops into the remaining k-i cups. We multiply these two quantities to get the total number of possibilities in this 2-step process. Finally, we sum all of the possible (i, k-i) partitions from 0 to k to consider all distributions, producing $\sum_{i=0}^{k} C(m+k-i-1, k-i) \cdot C(n+i-1, i)$ possibilities.

Since these two quantities count the same distribution of m+n scoops into k sundaes, they are identical. $\boxed{\text{W}}$

5. (20 points) Run Dijkstra's algorithm on the following graph to find the shortest path from a to z. Recall that at each stage of Dijkstra's algorithm, one vertex is chosen and given a permanent label which represents the length of the shortest path from a to that vertex. Write down the list of vertices in the order in which they are given permanent labels. Additionally, find the length of a shortest path from a to z.



1. $L(a)=0$
- $\ell(b)=1, \ell(e)=8$

2. $L(b)=1$
- $\ell(c)=4, \ell(e)=3$

3. $L(e)=3$
- $\ell(f)=10, \ell(g)=7$

4. $L(c)=4$
- $\ell(f)=6, \ell(d)=15$

5. $L(f)=6$
-

6. $L(g)=7$
- $\ell(d)=13, \ell(z)=16$

7. $L(d)=13$

8. $L(z)=16$

a, b, e, c, f, g, d, z

Shortest path length $=16$