Math 61, Lec 1
Winter 2016
Exam 1
1/25/16
Time Limit: 50 Minutes

Name (Print): ____________________

Name (Sign): ____________________

Discussion Section: ____________________

This exam contains 6 pages (including this cover page) and 5 problems. Check to see if any pages are missing.

You may *not* use books, notes, or any calculator on this exam.

If your answer contains a number that is impossible to simplify without the use of a calculator, such as e^3, $\ln(3)$ or $\sin(3)$, you may leave answers in terms of e, ln, or trig functions.

Partial credit will only be awarded to answers for which an explanation and/or work is shown.

Please attempt to organize your work in a reasonably neat and coherent way, in the space provided. If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	25	
2	18	
3	7	
4	25	
5	25	
Total:	100	

1. (25 points) Use mathematical induction to prove that $1+3+5+\cdots+(2n+1)=(n+1)^2$ for every integer $n \geq 0$.

Answer:
Base case: $n=0$. Then $2(0)+1=1=(0+1)^2$, so the base case holds.

Induction Step: Assume that the statement holds for some fixed integer $n \geq 0$. We wish to show this is true for $n+1$.
Then $1+3+5+\cdots+(2n+1)+(2(n+1)+1)=(n+1)^2+(2(n+1)+1)$ by induction hypothesis. But

$$\begin{aligned}(n+1)^2+(2(n+1)+1) &= (n^2+2n+1)+(2n+3)\\ &= n^2+4n+4\\ &= (n+2)^2\\ &= ((n+1)+1)^2\end{aligned}$$

Thus, $1+3+\cdots+(2n+1)+(2(n+1)+1)=((n+1)+1)^2$, so the statement is true for $n+1$.

Therefore, by induction, the statement is true for every integer $n \geq 0$.

2. (18 points) (a) Let X be a set with n elements and let Y be a set with m elements, where $m \geq n$. How many different one-to-one functions from X to Y are there?

Answer: $\dfrac{m!}{(m-n)!}$

(b) Let X be a set with n elements. How many different relations on X are there?

Answer: 2^{n^2}

(c) Let X be a set with n elements. How many different relations on X are there that are not reflexive?

Answer: $2^{n^2} - 2^{n^2-n}$

3. (7 points) Negate the following implication:
"If you are on the wait list, then you will be enrolled in the class."

Answer: "You are on the wait list, and you will not be enrolled in the class."

4. (25 points) Let X be a set. We define the *power set* $\mathcal{P}(X)$ to be the set of all subsets of X. Consider the sets $X = \{1, 2, 3\}$ and $Y = \{4, 5, 6\}$.
Define a function $f : \mathcal{P}(X) \times \mathcal{P}(Y) \to \mathcal{P}(X \cup Y)$ by $f((S, T)) = S \cup T$, for $S \in \mathcal{P}(X)$ and $T \in \mathcal{P}(Y)$. Prove that f is a bijection.

Answer: We want to prove that f is one-to-one and onto.

To show that f is one-to-one, we show that for any (S_1, T_1) and (S_2, T_2) in $\mathcal{P}(X) \times \mathcal{P}(Y)$, if $f((S_1, T_1)) = f((S_2, T_2))$, then $(S_1, T_1) = (S_2, T_2)$.
To this end, suppose that $f((S_1, T_1)) = f((S_2, T_2))$. Then $S_1 \cup T_1 = S_2 \cup T_2$. Since $X \cap Y = \emptyset$, we have that T_1 and T_2 contain no elements in X, and that S_1 and S_2 contain no elements in Y.
Therefore, $X \cap (S_1 \cup T_1) = S_1$, and $X \cap (S_2 \cup T_2) = S_2$. But $X \cap (S_1 \cup T_1) = X \cap (S_2 \cup T_2)$, so we conclude that $S_1 = S_2$.
Similarly, $Y \cap (S_1 \cup T_1) = T_1$, and $Y \cap (S_2 \cup T_2) = T_2$. But $Y \cap (S_1 \cup T_1) = Y \cap (S_2 \cup T_2)$, so we conclude that $T_1 = T_2$. Therefore, f is one-to-one.

To show that f is onto, we show that for any set $Z \in \mathcal{P}(X \cup Y)$, there exists an element $(S, T) \in \mathcal{P}(X) \times \mathcal{P}(Y)$ such that $f((S, T)) = Z$. To this end, let $Z \in \mathcal{P}(X \cup Y)$ be arbitrary.
Define S by $S = X \cap Z$, and define T by $T = Y \cap Z$.
Then $f((S, T)) = S \cup T = (X \cap Z) \cup (Y \cap Z) = (X \cup Y) \cap Z = Z$. Thus, f is onto.

5. (25 points) As in the previous problem, for any set X, we define the *power set* $\mathcal{P}(X)$ to be the set of all subsets of X. For example, if $X = \{1, 2\}$, then $\mathcal{P}(X) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$.

   Define a relation R on $\mathcal{P}(\mathbb{Z})$ by $(S, T) \in R$ if and only if $S \subseteq T$, for any sets S and T in $\mathcal{P}(\mathbb{Z})$. Prove that R is a partial order.

Answer: We want to show that R is reflexive, antisymmetric, and transitive.

R is reflexive: Let $S \in \mathcal{P}(\mathbb{Z})$ be arbitrary. Then $S \subseteq S$, so $(S, S) \in R$. Thus, R is reflexive.

R is antisymmetric: Let $S, T \in \mathcal{P}(\mathbb{Z})$ be arbitrary, and assume that $(S, T) \in R$ and that $(T, S) \in R$. Then $S \subseteq T$, and $T \subseteq S$. Then by properties of set inclusion, we have that $S = T$. Thus, R is antisymmetric.

R is transitive: Let $S, T, V \in \mathcal{P}(\mathbb{Z})$ be arbitrary, and assume that $(S, T) \in R$ and $(T, V) \in R$. Then $S \subseteq T$, and $T \subseteq V$. Then by properties of set inclusion, we have that $S \subseteq V$. Hence, $(S, V) \in R$. Thus, R is transitive.