Math 61, Lec 1
Winter 2016
Exam 2
2/22/16
Time Limit: 50 Minutes

Name (Print):

Name (Sign):

Discussion Section:

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This exam contains 6 pages (including this cover page) and 5 problems. Check to see if any pages are missing.

You may *not* use books, notes, or any calculator on this exam.

Unless otherwise stated in the problem, you may leave all answers in terms of $\binom{n}{k}$, $P(n,k)$, $k!$, or any sum, difference, product, or quotient of such symbols.

Partial credit will only be awarded to answers for which an explanation and/or work is shown.

Please attempt to organize your work in a reasonably neat and coherent way, in the space provided. If you need more space, use the back of the pages; clearly indicate when you have done this.

| Problem | Points | Score | |
|---|---|---|---|
| 1 | 24 | 18 | 6 |
| 2 | 16 | 16 | |
| 3 | 22 | 14 | 8 |
| 4 | 18 | 18 | |
| 5 | 20 | 20 | |
| Total: | 100 | 86 | |

1. (24 points) There are 999,999 natural numbers less than one million. We write any of them as a six digit number, including leading zeros. (For example, 001124 is how we write the number 1124).

   (a) How many of these numbers have all different digits?

   0, 1, 2, 3, 4, 5, 6, 7, 8, 9

   $= \underline{10} \cdot \underline{9} \cdot \underline{8} \cdot \underline{7} \cdot \underline{6} \cdot \underline{5}$ numbers w/ different digits.

   +8

   (b) How many of these numbers have exactly four distinct digits? (For example, 922433 is valid, but 922435 is not valid and 922444 is not valid).

   $\underline{10}\ \underline{9}\ \underline{8}\ \underline{7}$

   $10 \cdot 9 \cdot 8 \cdot 7$

   $P(10,4) \cdot \frac{6!}{2!2!}$

   +

   11234?,

   +4

   (c) How many of these numbers have digits that sum to 18?

   $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 18$

   $0 \leq x_i \leq 9$

   All $- \ x_i \geq 11$

   $y_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 7$

   $C(18+6-1, 18) - 6C(7+6-1, 7)$

   +6

2. (16 points) Solve the recurrence relation $A_n = 3A_{n-1} + 4A_{n-2}$, where $A_0 = 3$, $A_1 = 7$.

$$A_n - 3A_{n-1} - 4A_{n-2} = 0$$

$$t^2 - 3t - 4 = 0$$

$$(t-4)(t+1) = 0$$

$$t = 4, -1$$

$$A_n = b4^n + d(-1)^n$$

$$A_0 = b4^0 + d(-1)^0 = 3 \qquad A_1 = b4^1 + d(-1)^1 = 7$$

$$b + d = 3 \qquad 4b - d = 7$$

$$b = 3 - d \qquad 4(3-d) - d = 7$$

$$b = 3 - 1 \qquad 12 - 4d - d = 7$$

$$b = 2 \qquad -5d = -5$$

$$d = 1$$

$$A_n = 2(4^n) + (-1)^n$$

3. (22 points) Recall that a $k$-cycle is a cycle that includes $k$ edges. In this problem, you will prove Ramsey's theorem, which states that if $n \geq 6$ and we color each edge of $K_n$ either blue or red, then there must exist either a set of three blue edges that form a 3-cycle, or a set of three red edges that form a 3-cycle.
To this end, let $n \geq 6$ be arbitrary, and suppose every edge in $K_n$ is colored either blue or red. Let $v_1$ be a vertex in $K_n$.
Prove that at least three of the edges incident to $v_1$ are the same color.

For every vertex in $K_n$, the vertex has degree of $n-1$ by definition of complete graph.

So $\sum_{i=0}^{n} \delta(v) = 2|E|$

$|E| = \frac{1}{2}(n)(n-1)$.

$\sum_{i=0}^{n} \delta(v) = 2|E|$

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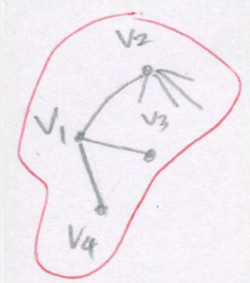
A vertex $v_1$ in $K_n$ will be of degree $n-1$, where $n \geq 1$, must also be in a n-cycle $C = \{v_1, e_1, v_2, e_2, \ldots v_n, e_n, v_1\}$ by definition of complete graph

$V_1$ has $n-1$ edges (pigeons) but each must be either red or blue (pigeonholes)
So there must be $\lceil \frac{n-1}{2} \rceil$ edges of same color. by PHP ✓
B/c $n \geq 6$, the # of edges of same edges must be at least $\lceil \frac{6-1}{2} \rceil = 3$

(b) In the previous part, you proved that at least three of the edges incident to $v_1$ are the same color. Without loss of generality, you may assume that color is blue. Suppose that $\{v_1, v_2\}$, $\{v_1, v_3\}$, and $\{v_1, v_4\}$ are blue edges. Prove that between these four vertices, there must exist either a blue 3-cycle or a red 3-cycle.

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$V_1$ must be in an n-cycle, $C = \{v_1, e_1, v_2, e_2, V_3, e_3, V_4, e_4, \ldots V_n, e_n, V_1\}$
by definition of complete graph $K_n$.
and for every $v, w \in V$ there is a path.
If we follow the one blue edge ie. $\{V_1, V_2\}$, $V_2$ then also must have 3 blue edges incident on it (including the one we came from). Of the 3, one must connect to $V_3$
false!
why? also false!
Then from $V_3$ we know there is another blue edge $\{V_1, V_3\}$
from which we can go back to $V_1$.
$V_1 \to V_2 \to V_3 \to V_1$

4. (18 points) Prove the combinatorial identity

$$\sum_{i=0}^{k} C(m+k-i-1, k-i) \cdot C(n+i-1, i) = C(m+n+k-1, k)$$

using a combinatorial argument. No more than half credit will be awarded to an algebraic proof. (Hint: Use Pirates and Gold.)

We will count the # of ways to put k balls into m+n boxes

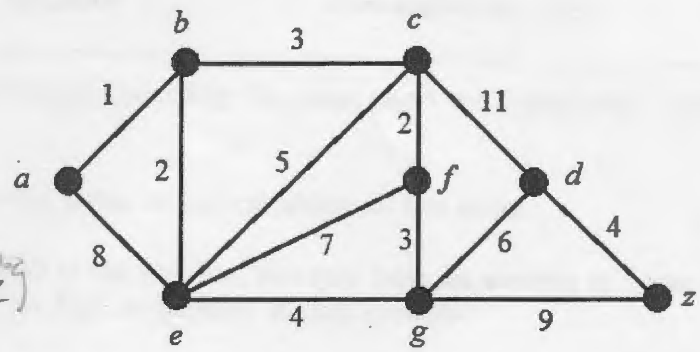
(1) We combine the boxes to get a total # of boxes, m+n. Then we divide the k balls into m+n groups. There are $C(k+m+n-1, k)$ ways to do this (ie using Pirates & Gold, k gold pieces, m+n pirates)

(2) This time, we will 1st choose how many balls go into the n boxes first. We call this value i for which $0 \le i \le k$. i can be as small as 1 ball or as large as k balls. So for some i # of balls, we can divide them into n boxes in $C(i+n-1, i)$ ways. Then there are k-i balls left to put into the m boxes. We can do this $C(k-i+m-1, k-i)$ ways to do this. So for some i, we use the MP and get $C(k-i+m-1, k-i) \cdot C(n+i-1, i)$ ways to put the k balls into the boxes.

And because i can range from $0 \le i \le k$, using the AP, we get $\sum_{i=0}^{k} C(m+k-i-1, k-i) \cdot C(n+i-1, i)$ ways to put k balls into m+n boxes.

Hence $\sum_{i=0}^{k} C(m+k-i-1, k-i) \cdot C(n+i-1, i) = C(m+n+k-1, k)$

5. (20 points) Run Dijkstra's algorithm on the following graph to find the shortest path from $a$ to $z$. Recall that at each stage of Dijkstra's algorithm, one vertex is chosen and given a permanent label which represents the length of the shortest path from $a$ to that vertex. Write down the list of vertices in the order in which they are given permanent labels. Additionally, find the length of a shortest path from $a$ to $z$.




$T=\{\cancel{a},\cancel{b},\cancel{c},\cancel{d},\cancel{e},\cancel{f},\cancel{g},z\}$

① $L(a)=0$
$l(b)=1$
$l(e)=8$

② $L(b)=1$
$l(c)=1+3=4$
$l(e)=\min\{8,1+2\}=3$

③ $L(e)=3$
$l(c)=\min\{4,3+5\}=4$
$l(f)=3+7=10$
$l(g)=3+4=7$

④ $L(c)=4$
$l(d)=4+11=15$
$l(f)=\min\{10,4+2\}=6$

⑤ $L(f)=6$
$l(g)=\min\{7,6+3\}=7$

⑥ $L(g)=7$
$l(d)=\min\{15,7+6\}=13$
$l(z)=7+9=16$

⑦ $L(d)=13$
$l(z)=\min\{16,13+4\}=16$

⑧ $L(z)=16$

Permanent labels =
a, b, e, c, f, g, d, z

shortest path length = 16
from a to z