Med: 78

Math 61, Lec 1
Winter 2016
Exam 1
1/25/16
Time Limit: 50 Minutes

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Discussio

This exam contains 6 pages (including this cover page) a pages are missing.

You may *not* use books, notes, or any calculator on this exam.

If your answer contains a number that is impossible to simplify without the use of a calculator, such as e^3, $\ln(3)$ or $\sin(3)$, you may leave answers in terms of e, ln, or trig functions.

Partial credit will only be awarded to answers for which an explanation and/or work is shown.

Please attempt to organize your work in a reasonably neat and coherent way, in the space provided. If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	7	7
2	18	17
3	25	22
4	25	25
5	25	25
Total:	100	96

1. (7 points) Negate the following implication:
"If you are on the wait list, then you will be enrolled in the class."

I am on the wait list, and I was not enrolled in the class.

p = on waitlist
q = enrolled in class

$p \wedge \neg q$

2. (18 points) (a) Let X be a set with n elements. How many different relations on X are there?

$1\,[a]$ $\quad$ $\begin{array}{c} \\ 1 \\ 2 \end{array}\begin{array}{cc} 1 & 2 \\ a & b \\ c & d \end{array}$ $\quad$ $\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{ccc} 1 & 2 & 3 \\ a & b & c \\ d & e & f \\ g & h & i \end{array}$

2^1 $\quad$ 2^4 $\quad$ 2^9 $\quad$ $1, 4, 9 \ldots n^2$

$\boxed{2^{n^2} \text{ relations.}}$

(b) Let X be a set with n elements. How many different relations on X are there that are not reflexive?

$[0]$ $\quad$ $\begin{array}{c} \\ 1 \\ 2 \end{array}\begin{array}{cc} 1 & 2 \\ 0 & a \\ a & 0 \end{array}$ $\quad$ $\begin{bmatrix} 0 & f & d \\ a & 0 & e \\ b & c & 0 \end{bmatrix}$ $\quad$ $\begin{bmatrix} 0 & a & b & c \\ g & 0 & d & e \\ h & i & 0 & f \\ j & k & l & 0 \end{bmatrix}$

2^0 $\quad$ 2^2 $\quad$ 2^6 $\quad$ 2^{12} $\quad$ $0, 2, 6, 12$

$n^2 - n$

$2^{n^2} - \boxed{2^{n^2-n} \text{ relations.}}$

(c) Let X be a set with n elements and let Y be a set with m elements, where $m \geq n$. How many different one-to-one functions from X to Y are there?

X $\quad$ Y

$\{1 \ldots n\}$ $\quad$ $\{1 \ldots m\}$

m

$m-1$

$m-2$

$m(m-n+1)$ $\quad$ m

$f : X \to Y$

$m \cdot (m-1) \cdots (m-(n+1))$

$\boxed{= \dfrac{m!}{(m-n)!}}$

for all $x \in X$:

choose 1 $y \in Y$ that has not been chosen,

3. (25 points) Use mathematical induction to prove that $1+3+5+\cdots+(2n+1)=(n+1)^2$ for every integer $n \geq 0$.

Base case:

$n=0$: $2(0)+1=1$ $(0+1)^2=1$ ✓

$1=1$

-3

$n \geq 1$: Assume $1+2+\ldots+(2n+1)=(n+1)^2$ for some $n \geq 0$.

$\underbrace{1+3+5+\ldots}_{(n+1)^2}+(2(n+1)+1)$ $((n+1)+1)^2$

$=(n+2)^2$

$=(n+1)^2+2n+3$

$=n^2+2n+1+2n+3$

$=n^2+4n+4$

$=(n+2)^2$

$=((n+1)+1)^2$

$(n+2)^2=(n+2)^n$ ✓

☺

4. (25 points) Let X be a set. We define the *power set* $\mathcal{P}(X)$ to be the set of all subsets of X. For example, if $X = \{1, 2\}$, then $\mathcal{P}(X) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$.

   Define a relation R on $\mathcal{P}(\mathbb{Z})$ by $(S, T) \in R$ if and only if $S \subseteq T$, for any sets S and T in $\mathcal{P}(\mathbb{Z})$. Prove that R is a partial order.

$U = \mathbb{Z}$

Reflexive: Let set $X \in P(\mathbb{Z})$ be arbitrary. $X \subseteq X$, so $(X,X) \in R$.

anti-symmetric: Let sets $X, Y \in P(\mathbb{Z})$ be arbitrary. Assume $(X,Y) \wedge (Y,X) \in R$. Then, $X \subseteq Y$, and $Y \subseteq X$. By definition of set equality, $X = Y$.

Transitive. Let sets $X, Y, Z \subseteq P(\mathbb{Z})$ be arbitrary. Assume $(X,Y) \wedge (Y,Z) \in R$. Then, $X \subseteq Y$, and $Y \subseteq Z$. Let $x \in X$ be arbitrary. Because $X \subseteq Y$, $x \in Y$. Then, because $Y \subseteq Z$, $x \in Z$. So, $\forall x \in X, x \in Z$. Therefore, $X \subseteq Z$, so $(X,Z) \in R$.

Reflexive, anti-symmetric, transitive $\Rightarrow$ Partial order

5. (25 points) As in the previous problem, for any set X, we define the *power set* $\mathcal{P}(X)$ to be the set of all subsets of X.
Consider the sets $X = \{1,2,3\}$ and $Y = \{4,5,6\}$.
Define a function $f : \mathcal{P}(X) \times \mathcal{P}(Y) \to \mathcal{P}(X \cup Y)$ by $f((S,T)) = S \cup T$, for $S \in \mathcal{P}(X)$ and $T \in \mathcal{P}(Y)$. Prove that f is a bijection.

1-1: Let $A, B \in P(X)$ and $C, D \in P(Y)$ be arbitrary, and assume $f(A, C) = f(B, D)$. Then, $A \cup C = B \cup D$. Let $x \in A$ be arbitrary. Because $x \in A$, $x \in A \cup C$, and $x \in B \cup D$. But, because $X \cap Y = \phi$, $x \notin D$, so $x \in B$. So, $A \subseteq B$. By choosing arbitrary $x \in B$ instead, by similar argument $B \subseteq A$. So, $A = B$. Using same logic for arbitrary $y \in C$, $y \in A \cup C \Rightarrow y \in B \cup D$, and because $y \notin B$, $y \in D$, so $C \subseteq D$, and also $D \subseteq C$. So, ~~because~~ $A = B \wedge C = D$, f is 1-1. ✓

Onto: Let $C \in P(X \cup Y)$ be arbitrary. Let $a \in C$ be arbitrary. By definition, $a \in X \vee a \in Y$. So, for some $A \in P(X)$, $B \in P(Y)$, $a \in A \vee a \in B$. So, $A \cup B \subseteq C$. Then, for some $b \in C$, for some $D \in P(X)$, $E \in P(Y)$, $b \in D \wedge b \notin E$. So, $C \subseteq D \cup E$. Because $a \in C$ and $b \in C$ are not mutually exclusive, $\exists A, D \in P(X) \wedge B, E \in P(Y)$ $A \cup B = D \cup E$. So, for $\forall C \in P(X \cup Y)\ \exists A \in P(X), B \in P(Y)$ $f(A,B) = C$. ✓

1-1 ∧ onto ⇒ Bijection $\boxed{\checkmark}$

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