Math 61, Lec 1
Winter 2016
Practice Problems
3/18/16
Time Limit: 180 Minutes

Name (Print): ____________________

Name (Sign): ____________________

Discussion Section: ____________________

These problems should give you a rough idea of the difficulty level of the final exam, although they are not meant to be an accurate representation of the length of the final exam.
You may *not* use books, notes, or any calculator on this exam.

As usual, unless otherwise stated in the problem, you may leave all answers in terms of $\binom{n}{k}$, $P(n, k)$, $k!$, or any sum, difference, product, or quotient of such symbols.

Partial credit will only be awarded to answers for which an explanation and/or work is shown.

Please attempt to organize your work in a reasonably neat and coherent way, in the space provided. If you need more space, use the back of the pages; clearly indicate when you have done this.

1. Prove that every tree is a planar graph. (Hint: since you cannot draw every possible tree, it is a bad idea to try to do this directly. Instead, prove this by contradiction, using Kuratowski's Theorem.)

Answer: Let T be a tree, and assume for sake of contradiction that T is not planar. Then by Kuratowski's Theorem, T contains a subgraph homeomorphic to K_5 or $K_{3,3}$. However, both K_5 and $K_{3,3}$ contain cycles, so since T has a subgraph homeomorphic to one of them, T must have a cycle. But trees have no cycles, so we get a contradiction.

2. Prove the combinatorial identity

$$n^2 = 2C(n, 2) + n$$

using a combinatorial argument.

Answer: Let $X = \{1, 2, \ldots, n\}$. We count the number of elements in $X \times X$ in two ways.

LHS: Every element in $X \times X$ is an ordered pair in which each coordinate comes from X. There are n ways to choose which element is the first coordinate, and n ways to choose which element is the second coordinate, so by multiplication principle, there are n^2 total elements in $X \times X$.

RHS: We partition the ordered pairs in $X \times X$ according to whether both coordinates are the same element, or different elements. If they are different elements, then two elements of X are represented. There are $C(n, 2)$ ways to choose which two elements of X are represented. However, the order in an ordered pair matters, so we must choose which of the two elements is the first coordinate. There are 2 ways to choose which of the two elements is the first coordinate. Thus, there are $2C(n, 2)$ ordered pairs in $X \times X$ in which the coordinates are different elements. If the coordinates are the same element, we just choose one element from the n elements in X. There are n ways to choose the repeated element. Thus, there are n ordered pairs in $X \times X$ in which the coordinates are the same. Hence, there are $2C(n, 2) + n$ total elements in $X \times X$.

Thus, $n^2 = 2C(n, 2) + n$.

3. Now, prove the same combinatorial identity,

$$n^2 = 2C(n, 2) + n$$

using induction.

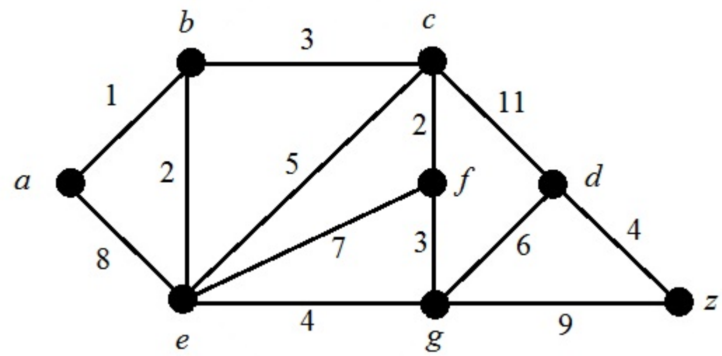
Answer: We proceed by induction. Base case: $n = 2$ (if you used $n = 0$ or $n = 1$, those are both fine). Then $2^2 = 4$, and $C(2, 2) = 1$, so $2C(2, 2) + 2 = 4$. Thus, the base case holds.

Induction step: Assume that for some fixed n, we have that $n^2 = 2C(n, 2) + n$. We want to show that it holds for $n + 1$.
But $(n+1)^2 = n^2+2n+1$, and by the induction hypothesis, $n^2+2n+1 = 2C(n, 2)+n+2n+1$. Thus,

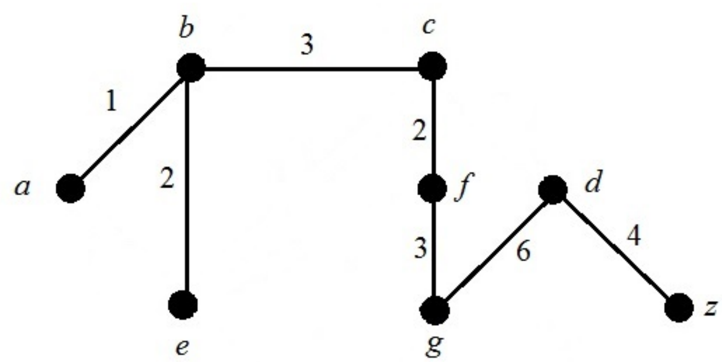
$$\begin{aligned}
(n+1)^2 &= n^2 + 2n + 1 \\
&= 2C(n, 2) + n + 2n + 1 \\
&= 2\left(\frac{n(n-1)}{2}\right) + 3n + 1 \\
&= n(n-1) + 3n + 1 \\
&= n^2 - n + 3n + 1 \\
&= n^2 + n + n + 1 \\
&= (n + 1)n + n + 1 \\
&= 2C(n + 1, 2) + n + 1.
\end{aligned}$$

Therefore, $(n + 1)^2 = 2C(n + 1, 2) + (n + 1)$, so the statement is true for $n + 1$. Therefore, by induction, the statement is true for all n.

4. Run Prim's algorithm on the following graph to find a minimal spanning tree, starting at a. Write down the list of vertices in the order in which they are added to the spanning tree with Prim's, and draw the minimal spanning tree.



Answer: a, b, e, c, f, g, d, z



5. Let $\mathcal{G}$ be the set of all simple graphs. We define a relation R on $\mathcal{G}$ by $(G, H) \in R$ if and only if G and H have the same number of vertices of degree 3.

(a) Show that R is an equivalence relation.

Answer: We must show that R is reflexive, symmetric, and transitive.

(i) R is reflexive: Let $G \in \mathcal{G}$ be arbitrary. Then G has the same number of vertices of degree 3 as itself, so $(G, G) \in R$.

(ii) R is symmetric: Let $G, H \in \mathcal{G}$ and assume that $(G, H) \in R$. Then G and H have the same number of vertices of degree 3, so H and G have the same number of vertices of degree 3, so $(H, G) \in R$.

(iii) R is transitive: Let $G, H, I \in \mathcal{G}$ and assume that $(G, H) \in R$ and $(H, I) \in R$. Then G has the same number of vertices of degree 3 as H, and H has the same number of vertices of degree 3 as I, so G must have the same number of vertices of degree 3 as I. Thus, $(G, I) \in R$.

Thus, R is an equivalence relation.

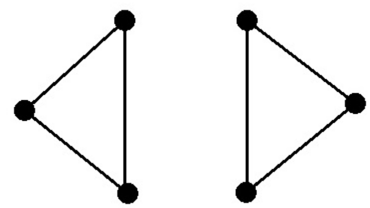
(b) For what values of m and n do we have that $(K_4, K_{m,n}) \in R$?

Answer: K_4 has 4 vertices of degree 3, so we want to find m and n such that $K_{m,n}$ has exactly 4 vertices of degree 3. But $K_{m,n}$ has m vertices of degree n and n vertices of degree m. Thus, $K_{m,n}$ has 4 vertices of degree 3 if and only if $m = 4$ and $n = 3$ or $m = 3$ and $n = 4$. Thus, $(K_4, K_{4,3}) \in R$ and $(K_4, K_{3,4}) \in R$.

6. As in the previous problem, let $\mathcal{G}$ be the set of all simple graphs. Define a function $f : \mathcal{G} \to \mathcal{G}$ as follows: for a graph $G = (V, E)$, define $f(G)$ to be the simple graph (V', E'), where $V' = V$ and for every $v, w \in V$, we have that the edge $(v, w) \in E'$ if and only if $(v, w) \notin E$. (Aside: $f(G)$ is called the *complement* of G).

   (a) Draw $f(K_{3,3})$

   Answer:



   (b) Prove that f is one-to-one in the following sense: if $f(G) \cong f(H)$, then $G \cong H$.

   Answer: Let $G, H \in \mathcal{G}$ be arbitrary, and suppose $f(G) \cong f(H)$. Write $G = (V_G, E_G)$ and $H = (V_H, E_H)$. Then $f(G)$ also has vertex set V_G and $f(H)$ also has vertex set V_H. Since $f(G)$ is isomorphic to $f(H)$, there exists a bijection $h : V_G \to V_H$ such that v and w are adjacent in $f(G)$ if and only if $h(v)$ and $h(w)$ are adjacent in $f(H)$. But v and w are adjacent in $f(G)$ if and only if v and w are not adjacent in G. Also, $h(v)$ and $h(w)$ are adjacent in $f(H)$ if and only if $h(v)$ and $h(w)$ are not adjacent in H. Therefore, h is a bijection from the vertex set of G to the vertex set in H such that v and w are adjacent in G if and only if $h(v)$ and $h(w)$ are adjacent in H. Hence, $G \cong H$.

7. Suppose that $n, m > 3$.

   (a) If $2 < k < n$, how many different simple cycles of length k are there in K_n?

   Answer: There are $\dfrac{n!}{(n-k)!}$ simple paths of length $k-1$, and then the last step must be the edge back to the starting point to make it a cycle. However, this overcounts, since any of the k vertices could be the starting point, and we can traverse the cycle in either direction.
   Thus, the answer is $\dfrac{n!}{2k(n-k)!}$.

   (b) If $1 < k < \min\{m, n\}$, how many different simple cycles of length $2k$ are there in $K_{m,n}$?

   Answer: Such a cycle must hit k of the m vertices on the left, and k of the n vertices on the right. But by similar reasoning as in (a), we must divide by $2k$, since any vertex could be the starting point.
   Thus, the answer is $\dfrac{m! \cdot n!}{2k(m-k)!(n-k)!}$

   (c) How many different cycles of length 3 are there in $K_{m,n}$?

   Answer: 0.

8. (a) For which values of n and m does K_n contain $K_{m,m}$ as a subgraph?

Answer: For $m \leq n/2$.

(b) For which values of n and m does $K_{m,m}$ contain K_n as a subgraph?

Answer: Only when $n \leq 2$ and $m \geq n - 1$.

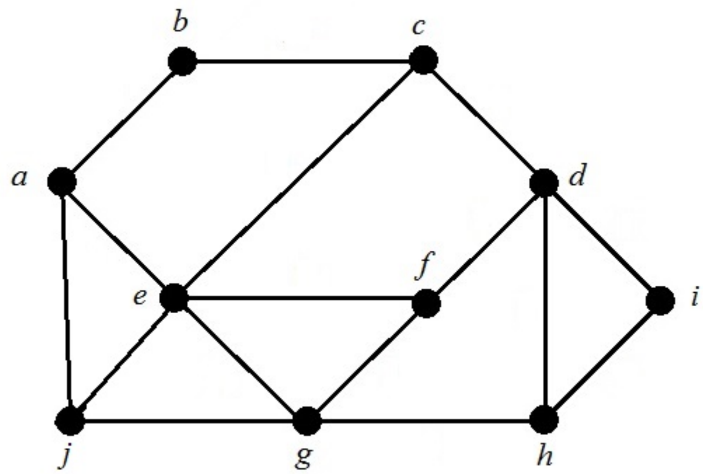
9. (a) Write the adjacency matrix for the 2-cube.

Answer: $\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$

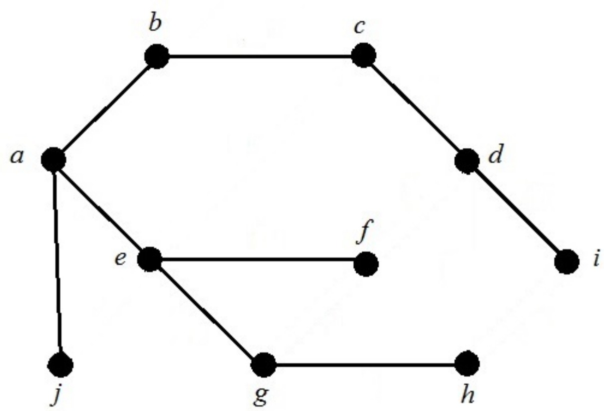
(b) Using the previous part, find the matrix that gives the number of paths of length 2 from any vertex to any other vertex in the 2-cube.

Answer: $\begin{pmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{pmatrix}$

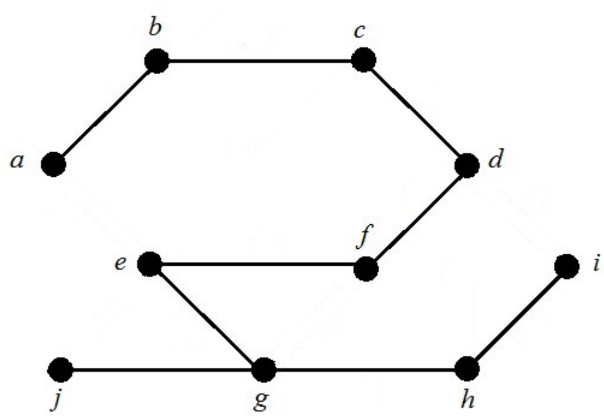
10. Consider the following graph. Draw the spanning trees that result from performing breadth-first search and depth-first search, both with respect to the vertex ordering $abcdefghij$.



Answer: Bread-first search tree is



Depth-first search tree is



11. Prove that the property "has distinct vertices v, w, and x such that every path from v to w contains x" is an invariant property.

Answer: Suppose $G_1 \cong G_2$ and that G_1 has distinct vertices v, w, and x such that every path from v to w contains x. Let $f : V_1 \to V_2$, $g : E_1 \to E_2$ be an isomorphism. Consider the vertices $f(v)$, $f(w)$, and $f(x)$. Since f is one-to-one, these vertices are distinct in G_2. We claim that every path from $f(v)$ to $f(w)$ contains $f(x)$.

To this end, let $P = (f(v), e_1, v_1, e_2, v_2, \ldots, v_{n-1}, e_n, f(w))$ be an arbitrary path from $f(v)$ to $f(w)$. Then for each i, $g^{-1}(e_i)$ is incident on $f^{-1}(v_{i-1})$ and $f^{-1}(v_i)$, so that:

$$(v, g^{-1}(e_1), f^{-1}(v_1), \ldots f^{-1}(v_{n-1}), g^{-1}(e_n), w)$$

is a path in G_1 from v to w. Then by the property, this path contains x, so there exists j such that $f^{-1}(v_j) = x$. But then $v_j = f(x)$, so $f(x)$ is contained in the path P.

12. Suppose you have four coins that are identical in appearance, but either one of the coins, or none of the coins, is either heavier or lighter than the others. Additionally, you have one extra coin that you know is the correct weight. You have only a pan balance. Is it possible to identify the bad coin, or prove that there is no bad coin, in at most two weighings? If so, draw a decision tree. If not, prove why not.

Answer: The picture was hard to draw in a manner that made it embeddable in this, so I'll explain the method. Suppose that C_1, C_2, C_3, and C_4 are the mystery coins, and C is the extra coin that you know is correct.

First, weigh C_1C_2 vs C_3C. If they are even, weigh C_4 vs C. If C_4 is heavier or lighter, the balance will be uneven. If they are even, it means there is no bad coin.

Otherwise, weigh C_1C_3 vs C_4C. If C_1C_2 is heavier than C_3C and C_1C_3 is heavier than C_4C, then C_1 is heavy. If C_1C_2 is heavier than C_3C and C_1C_3 is lighter than C_4C, then C_3 is light. If C_1C_2 is heavier than C_3C and C_1C_3 is even with C_4C, then C_2 is heavy. Apply similar reasoning to the three cases on the other side.

It is interesting to note that without the extra coin, this is impossible to do in two weighings, for the following reason. The first weighing has to either be 1 vs 1 or 2 vs 2. If the first weighing is 1 vs 1, then there are 5 different outcomes that are possible if the first weighing is even, and 5 outcomes can't be sorted out by one weighing. If the first weighing is 2 vs 2, then if one side is heavy there are 4 outcomes that are possible, and they can't be sorted out by one weighing. Thus, you can't do it in two weighings.

13. Although it is usually difficult to show that a graph has a Hamiltonian cycle, we also saw that the complete graph K_n always has a Hamiltonian cycle. In fact, if a graph consists only of vertices of high degree, we are guaranteed to have a Hamiltonian cycle. Precisely, we have the following:
Dirac's Theorem: Let $G = (V, E)$ be a simple graph with n vertices. If $\delta(v) \geq \frac{n}{2}$ for all $v \in V$, then G has a Hamiltonian cycle.

In this problem and the next problem, we will prove Dirac's Theorem. Suppose G is a simple graph with n vertices such that $\delta(v) \geq \frac{n}{2}$ for all $v \in V$. We first show that if x and y are distinct vertices in G such that (x, y) is not an edge in G, then G has a Hamiltonian cycle if and only if the graph resulting from adding the edge (x, y) to G has a Hamiltonian cycle.

The forward direction follows easily from the definition of a Hamiltonian cycle. We show the reverse direction. To that end, suppose that the graph resulting from adding the edge $e = (x, y)$ to G has a Hamiltonian cycle. If this cycle does not use the edge (x, y), then we're done. If the cycle does use (x, y), write the cycle as $(x, e, y, e_1, v_1, e_2, \ldots, v_{n-2}, e_{n-1}, x)$.
We count how many of the other vertices have edges to x and y, and we do this by counting indices.
Let $T = \{i \in \{1, 2, \ldots n-3\} : (v_i, x) \in E\}$ and let $S = \{i \in \{1, 2, \ldots n-3\} : (v_{i+1}, y) \in E\}$.
Then $|T| = \delta(x) - 1$ and $|S| = \delta(y) - 1$.

(a) Using the inclusion/exclusion principle, show that $|S \cap T| \geq 1$.

Answer: By inclusion/exclusion, $|S \cup T| = |S| + |T| - |S \cap T|$, and therefore, $|S \cap T| = |S| + |T| - |S \cup T|$.

But $|S\| \geq \frac{n}{2} - 1$, and $|T| \geq \frac{n}{2} - 1$, and we have that $|S \cup T| \leq n - 3$. Therefore,

$$\begin{aligned}
|S \cap T| &= |S| + |T| - |S \cup T| \\
&\geq \frac{n}{2} - 1 + \frac{n}{2} - 1 - (n-3) \\
&= n - 2 - (n-3) \\
&= 1.
\end{aligned}$$

Hence, $|S \cap T| \geq 1$.

(b) In the last part, you showed that $S \cap T \neq \emptyset$. Therefore, there exists i such that $(v_i, x) \in E$ and $(v_{i+1}, y) \in E$. Conclude that G has a Hamiltonian cycle that doesn't use the edge (x, y).

Answer: Let $\hat{e}$ be the edge (v_i, x) and let $\bar{e}$ be the edge (v_{i+1}, y).
We consider the cycle $(y, e_1, v_1, \ldots, v_{i-1}, e_i, v_i, \hat{e}, x, e_{n-1}, v_{n-2}, \ldots e_{i+2}, v_{i+1}, \bar{e}, y)$. Then this is a Hamiltonian cycle that doesn't use the edge (x, y).

14. Now we use the previous problem to prove Dirac's Theorem. So far, you have shown that for any simple graph G with n vertices in which every vertex has degree at least $n/2$, then G does not have a Hamiltonian cycle if and only if G will still not have a Hamiltonian cycle if we connect one pair of non-adjacent edges.

    To prove Dirac's Theorem, we assume for sake of contradiction that there exists a simple graph G with n vertices in which every vertex has degree at least $n/2$ that doesn't have a Hamiltonian cycle. Then since such a graph exists, we let H be such a graph with the maximal amount of edges. But since every complete graph has a Hamiltonian cycle, we know that H has some pair of vertices without an edge between them.

    Use the result of the previous problem in order to get a contradiction.

    Answer: As stated, we assume that H is such a graph without a Hamiltonian cycle with the maximal amount of edges. Since H can't be complete, there must be a pair of vertices x and y such that $(x, y) \notin E$. But by the previous problem, H doesn't have a Hamiltonian cycle if and only if H still doesn't have a Hamiltonian cycle when we add the edge (x, y). But then when we add this edge to H, we get a new graph with more edges than H that still doesn't have a Hamiltonian cycle. This contradicts the assumption that H had the maximal amount of edges.