1. Let A be a set of 4 elements and B a set of 3 elements

   (a) (2 pts) How many elements in the set $A \times B$?

   $|A \times B| = |A| \times |B| = 4 \times 3 = \boxed{12}$

   (b) (2 pts) How many 4-element subset does $A \times B$ have?

   $C_{12}^{4} = \frac{12 \times 11 \times 10 \times 9}{4 \times 3 \times 2 \times 1} = \boxed{495}$

   (c) (4 pts) How many onto functions are there from A to B?

   There are 3^4 function from A to B

   ① if 4 elements in A are mapped to same element in B : $C_3^1 = 3$

(c) (4 pts) How many onto functions are there from A to B?

There are 3^4 function from A to B

① if 4 elements in A are mapped to same element in B : $C_3^1 = 3$

② if 4 element in A are mapped to ≤2 elements in B

1) three in A to 1 in B, one in A to 1 in B : $C_3^1 \cdot C_2^1 \cdot C_4^1 = 12$

2) 2 in A to 1 in B, 2 in A to 1 in B: $C_3^1 \cdot C_2^1 \cdot C_4^2 = 36$

onto function : 4 elements in A are mapped to 3 elements in B

$3^4 - (12+36) + 3 = \boxed{36}$

(d) (2 pts) How many one-to-one functions are there from B to A?

A 4

B 3

$C_4^1 \cdot C_3^1 \cdot C_2^1 = \boxed{24}$

2. (a) (4 pts) For propositions p, q, and r use the truth table to prove

$$p \to (q \to r) \equiv \neg p \vee (\neg r \to \neg q)$$

p	q	r	$q \to r$	$p \to (q \to r)$	$\neg p$	$\neg r$	$\neg q$	$\neg r \to \neg q$	$\neg p \vee (\neg r \to \neg q)$
T	T	T	T	T	F	F	F	T	T
T	T	F	F	F	F	T	F	F	F
T	F	T	T	T	F	F	T	T	T
F	T	T	T	T	T	F	F	T	T
T	F	F	T	T	F	T	T	T	T
F	T	F	F	T	T	T	F	F	T
F	F	T	T	T	T	F	T	T	T
F	F	F	T	T	T	T	T	T	T

(b) (2 pts) State the negation of the the proposition: $(\forall n \in \mathbb{N})\sqrt{n}$ is either an integer or an irrational number.

(b) (2 pts) State the negation of the the proposition: $(\forall n \in \mathbb{N})\sqrt{n}$ is either an integer or an irrational number.

$(\exists n \in \mathbb{N})\sqrt{n}$ is not an integer and $\sqrt{n}$ is not an irrational number.

In words: There exists an integer n, n is an element of the set of non-negative integers, such that $\sqrt{n}$ is not an integer and $\sqrt{n}$ is not an irrational number.

(c) (4 pts) State the negation of the proposition: $(\exists n \in \mathbb{N})2^n \leqslant n$. Prove that this negation is true. Hint: use mathematical induction.

negation: $(\forall n \in \mathbb{N})\ 2^n > n$

proof: ① for $n=0$, $2^0 = 1 > 0$

for $n=1$, $2^1 = 2 > 1$

② Assume this statement is true for $n=k$, i.e $2^k > k$ for $k \in \mathbb{N}^+$

then by assumption, $2^k > k$

$\therefore\ 2 \cdot 2^k > 2 \cdot k$

$\therefore\ 2^{k+1} > k+k \geqslant k+1$

$\therefore\ 2^{k+1} > k+1$

$\because$ this statement is true for $n=1$, and $n=k$ $(k \in \mathbb{N}^+)$ true implies $n=k+1$ $(k \in \mathbb{N}^+)$ is true

$\therefore\ (\forall n \in \mathbb{N}^+),\ 2^n > n$

this statement is also true for $n=0$

$\therefore\ (\forall n \in \mathbb{N}),\ 2^n > n$

3. Solve the following recurrence relations.

(a) (3 pts) $a_n = 3a_{n-1} - a_{n-2}, a_1 = 1, a_2 = 3.$

① $f(x) = x^2 - 3x + 1 = 0$

$\Delta = b^2 - 4ac = 9 - 4 = 5 > 0$

$\therefore\ x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{3 \pm \sqrt{5}}{2}$

$\therefore\ r_1 = \frac{3+\sqrt{5}}{2} \quad r_2 = \frac{3-\sqrt{5}}{2}$

$\therefore\ a_n = c_1\left(\frac{3+\sqrt{5}}{2}\right)^n + c_2\left(\frac{3-\sqrt{5}}{2}\right)^n$

$\therefore r_1 = \frac{3+\sqrt{5}}{2} \quad r_2 = \frac{3-\sqrt{5}}{2}$

$\therefore a_n = C_1\left(\frac{3+\sqrt{5}}{2}\right)^n + C_2\left(\frac{3-\sqrt{5}}{2}\right)^n$

② $\begin{cases} C_1\left(\frac{3+\sqrt{5}}{2}\right)^1 + C_2\left(\frac{3-\sqrt{5}}{2}\right)^1 = 1 \\ C_1\left(\frac{3+\sqrt{5}}{2}\right)^2 + C_2\left(\frac{3-\sqrt{5}}{2}\right)^2 = 3 \end{cases}$

$\therefore \begin{cases} C_1\left(\frac{3+\sqrt{5}}{2}\right) + C_2\left(\frac{3-\sqrt{5}}{2}\right) = 1 \\ C_1\left(\frac{7+3\sqrt{5}}{2}\right) + C_2\left(\frac{7-3\sqrt{5}}{2}\right) = 3 \end{cases} \Rightarrow \begin{cases} C_1 = \frac{1}{\sqrt{5}} \\ C_2 = -\frac{1}{\sqrt{5}} \end{cases}$

$\boxed{\therefore a_n = \frac{(3+\sqrt{5})^n - (3-\sqrt{5})^n}{\sqrt{5}\cdot 2^n}}$

(b) (3 pts) $a_n = 4a_{n-1} - 4a_{n-2}, a_1 = 3, a_2 = 5.$

① $f(x) = x^2 - 4x + 4 = 0 \quad \therefore x_1 = x_2 = 2$

$\therefore a_n = C_1 \cdot 2^n + C_2 \cdot n \cdot 2^n$

② $\begin{cases} a_1 = 3 = C_1 \cdot 2 + C_2 \cdot 1 \cdot 2 \\ a_2 = 5 = C_1 \cdot 2^2 + C_2 \cdot 2 \cdot 2^2 \end{cases}$

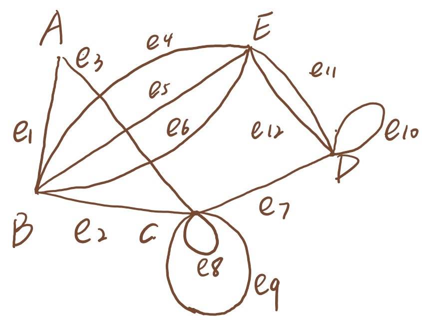
$\therefore \begin{cases} 2C_1 + 2C_2 = 3 \\ 4C_1 + 8C_2 = 5 \end{cases} \Rightarrow \begin{cases} C_1 = \frac{7}{4} \\ C_2 = -\frac{1}{4} \end{cases}$

$\boxed{\therefore a_n = \frac{7}{4} \cdot 2^n - \frac{1}{4} \cdot n \cdot 2^n}$

4. (a) (4 pts) Draw the graph represented by the adjacency matrix:

4. (a) (4 pts) Draw the graph represented by the adjacency matrix:

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{matrix} A & B & C & D & E \end{matrix} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 3 \\ 1 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 3 & 0 & 2 & 0 \end{bmatrix} \end{array}$$



(b) (4 pts) Please label the edges for the graph obtained in part (a). Find the corresponding incidence matrix for the graph in part (a) with your labeled edges.

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}
A	1	0	1	0	0	0	0	0	0	0	0	0
B	1	1	0	1	1	1	0	0	0	0	0	0
C	0	1	1	0	0	0	1	1	1	0	0	0
D	0	0	0	0	0	0	1	0	0	1	1	1
E	0	0	0	1	1	1	0	0	0	0	1	1

5. (5 pts) Let $S = \mathbb{R} \setminus \mathbb{Q}$. Consider the following relation $\sim$ on S: $x \sim y$ if and only if $\frac{x}{y} \in \mathbb{Q}$. Is $\sim$ an equivalence relation? Prove your conclusion.

① $\forall x \in S$

$\because \frac{x}{x} = 1$, $1 \in \mathbb{Q}$

$\therefore x \sim x$ for all $x \in S$

$\therefore$ relation is reflective

③ $\forall x \in S$, $\forall y \in S$

if $x \sim y$, then $\frac{x}{y} \in \mathbb{Q}$

$\because 0 \notin S$

$\therefore x \neq 0$, $y \neq 0$

$\because \frac{x}{y} \in \mathbb{Q}$ $\therefore \exists p \in \mathbb{Z}, q \in \mathbb{Z}$

s.t. $\frac{x}{y} = \frac{p}{q}$

$\therefore \frac{y}{x} = \frac{q}{p}$

$\therefore \frac{y}{x} \in \mathbb{Q}$

$\therefore y \sim x$

$\therefore$ relation is symmetric

$\boxed{\therefore \text{ relation is equivalent}}$

② let $x, y, z \in S$

if $x \sim y$, $y \sim z$

then $\frac{x}{y} \in \mathbb{Q}$, $\frac{y}{z} \in \mathbb{Q}$

$\exists a, b, c, d \in \mathbb{Z}$ s.t. $\frac{x}{y} = \frac{a}{b}$, $\frac{y}{z} = \frac{c}{d}$

$\therefore \frac{x}{z} = \frac{x}{y} \cdot \frac{y}{z} = \frac{a}{b} \cdot \frac{c}{d}$

$\therefore \frac{x}{z} \in \mathbb{Q}$

$\therefore x \sim z$

$\therefore$ relation is transitive

6. (10 pts) Consider the weighted graph in Figure 1. Find the lengths of a shortest path from vertex E to other vertexes by Dijkstra's algorithm.

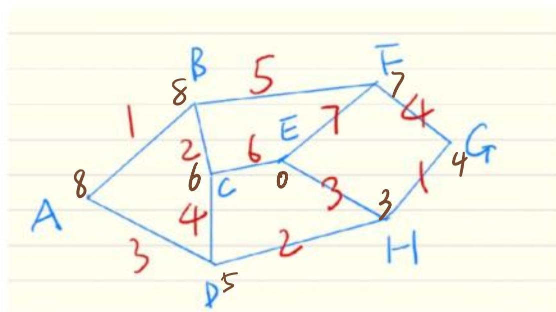


Figure 1: Graph for Question 6.

Sol=

Sol: ① $L(E)=0$

$L(A) = L(B) = L(C) = L(D) = L(F) = L(G) = L(H) = \infty$

$T = \{A, B, C, D, E, F, G, H\}$.

$L(E)=0$ is the minimal $\therefore$ choose E

② $L(C)=6$, $L(F)=7$, $L(H)=3$, $L(A)=L(B)=L(D)=L(G)=\infty$

$T=\{A, B, C, D, F, G, H\}$

$L(H)=3$ is the minimal $\therefore$ choose H, E to H $= 3$

③ $L(C)=6$, $L(F)=7$, $L(D)=5$, $L(G)=4$, $L(A)=L(B)=\infty$

$T=\{A, B, C, D, F, G\}$, $L(G)=4$ is minimal $\therefore$ choose G E to G $= 4$

④ G to F $= 4+4=8>7$ $\therefore$ $L(A)=L(B)=\infty$, $L(C)=6$, $L(D)=5$, $L(F)=7$

$T=\{A, B, C, D, F\}$, $L(D)=5$ is minimal $\therefore$ choose D E to D $=5$

⑤ $L(A)=8$, $L(B)=\infty$, $L(C)=6$, $L(F)=7$

$T=\{A, B, C, F\}$, $L(C)=6$ is minimal $\therefore$ choose C, E to C $=6$

$T=\{A,B,C,F\}$, $L(C)=6$ is minimal ∴ choose C

⑥ $L(A)=L(B)=8$, $L(F)=7$

$T=\{A,B,F\}$, $L(F)=7$ is minimal ∴ choose F, E to F = 7

⑦ $L(A)=L(B)=8$, $T=\{A,B\}$

no minimal ∴ choose either A or B

let's choose A, E to A = 8

⑧ A to B = 8+1 = 9 > 8 ∴ E to B minimal = 8

∴ (E to A) = 8, (E to B) = 8, (E to C) = 6, (E to D) = 5, (E to E) = 0

(E to F) = 7, (E to G) = 4, (E to F) = 3

7. Use Prim's Algorithm and Kruskal's Algorithm to find the minimal spanning tree for the weighted graph in Figure 2.(Please write down the order of the edges that you add to your spanning tree).

(a) (5 pts) The results by using Prim's Algorithm (consider the alphabetical order).

(b) (5 pts) The results by using Kruskal's Algorithm

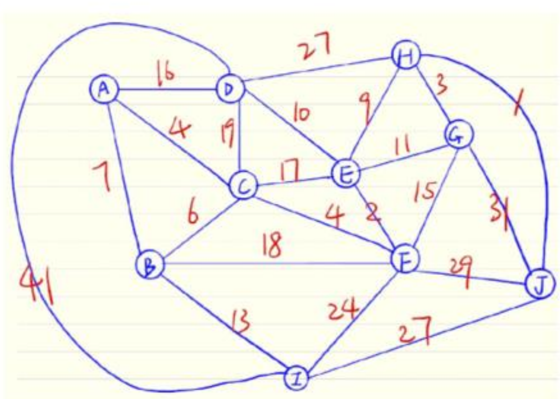


Figure 2: Graph for Question 7

(a) (A,C) → (C,F) → (F,E) → (C,B) → (E,H) → (H,J)

→ (H,G) → (E,D) → (B,I)

(b) $(H,J) \to (E,F) \to (H,G) \to (A,C) \to (C,F) \to (C,B)$
$\to (E,H) \to (E,D) \to (B,I)$

8. Determine whether each graph is planar. If the graph is planar, redraw it so that no edges cross; otherwise, find a subgraph homeomorphic to either K_5 or $K_{3,3}$.

   (a) (10 pts) See Figure in 3.

   (b) (5 pts) See Figure in 4

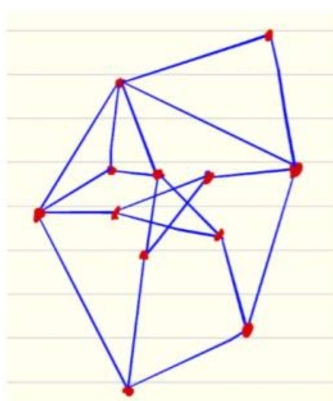
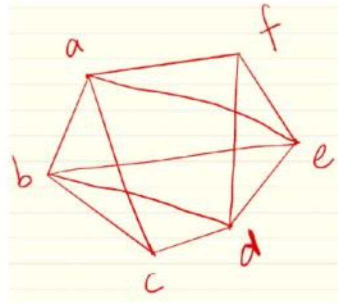


Figure 3: Graph for Question 8(a)

8(b)

a

Figure 3: Graph for Question 8(a)



8(b)

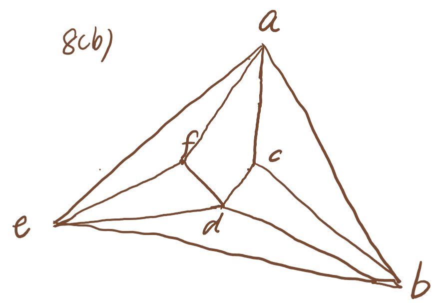
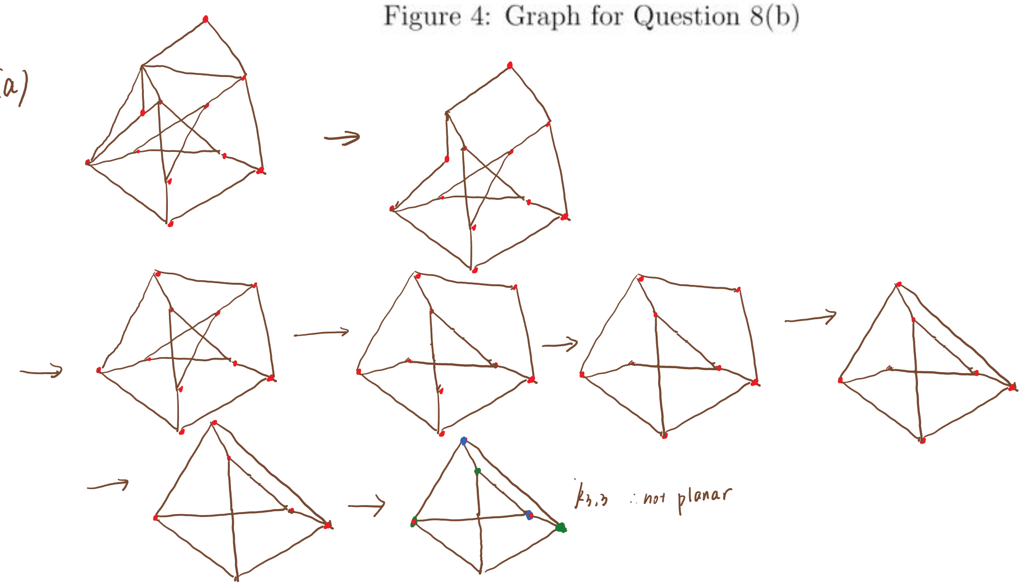


Figure 4: Graph for Question 8(b)

8(a)



9. (a) (10 pts) How many 11-digit binary sequences (i.e., each digit is either a 0 or a 1) do not contain two consecutive 0's?

all 1: 1

ten 1s, one 0 : $C_{11}^{1} = 11$

nine 1s, two 0s : $C_{10}^{2} = 45$

eight 1s, three 0s : $C_{9}^{3} = 84$

seven 1s, four 0s : $C_{8}^{4} = 70$

six 1s, five 0s : $C_{7}^{5} = C_{7}^{2} = 21$

five 1s, six 0s $C_{6}^{6} = 1$

$\therefore 1 + 11 + 45 + 84 + 70 + 21 + 1$

$= 233$

(b) (4 pts) A bakery produces 4 kinds of cookies (suppose there are infinitely many of each). A person wants to buy 6 cookies. Find the number of ways the person can buy 6 cookies.

o || o o | o o o

There are 4 kinds of cookies → divide 6 cookies with 3 splits, correspond to the graph (the graph is an example)

∴ # ways to buy 6 cookies = # ways to put 3 splits $= C_9^3 = \boxed{84}$

10. (a) (6 pts) Choose 150 integers from this list $\{1, 2, \cdots, 298\}$, prove that there are two integers n_1, n_2 such that $n_1 | n_2$ or $n_2 | n_1$.

Sol: for each integer $n \in \{1, 2, \cdots, 298\}$, n can be written as $n = t \cdot 2^k$, $t \in \mathbb{N}^+$, $k \in \mathbb{N}$, k is the largest possible integer

Then if $t \% 2 = 0$, then we can add 1 to k

we are assigning 150 pigeons to 149 holes, so there must be 2 pigeons in the same hole.

i.e $\exists\, n_1 = t_1 \cdot 2^{k_1}$, $n_2 = t_2 \cdot 2^{k_2}$ s.t $t_1 = t_2$

Then, for $n_1 = t_1 \cdot 2^{k_1}$, $n_2 = t_2 \cdot 2^{k_2}$, $t_1 = t_2$

if $k_1 > k_2$, then $n_2 | n_1$

k is the largest possible integer

Then, if $t\%2=0$, then we can add 1 to k

Thus, when $k\in\mathbb{N}$, k is the largest integer

$t\%2=1$

$\forall n_1, n_2 \in \{1,2,\cdots,298\}$

let $n_1 = t_1 \cdot 2^{k_1}$, $n_2 = t_2 \cdot 2^{k_2}$

For t_1, t_2, there are 149 possible odd number $\in\{1,2,\cdots,298\}$, but we need to choose 150 numbers. By pigeonhole principle 1st version,

Then, for $n_1 = t_1\cdot 2^{k_1}$, $n_2 = t_2 \cdot 2^{k_2}$, $t_1 = t_2$

if $k_1 > k_2$, then $n_2 | n_1$

if $k_1 < k_2$, then $n_1 | n_2$

$\therefore \exists n_1, n_2$ s.t $n_1 | n_2$ or $n_2 | n_1$

$\therefore$ the statement is true.

(b) (6 pts) Let $n_1, n_2, \cdots, n_{201}$ be integers. Prove there exist three integers $n_i, n_j, n_k \in \{n_1, n_2, \cdots, n_{201}\}$ such that 100 can divide the differences between any two of them.

Sol: $100 | (n_p - n_q)$ is equivalent to $(n_p - n_q)\%100 = 0$

let $n_p = 100m + A$, $n_q = 100n + B$

$n_p\%100 - n_q\%100 = A - B$

$= (n_p - n_q)\%100 = (100m + A - (100n + B))\%100 = A - B$

$\therefore$ if $(n_p - n_q)\%100 = 0$, then $n_p\%100 = n_q\%100$

for $n_i\%100$, there are 0-99 100 possible reminders

for $n_j\%100$, there are 0-99 100 possible reminders

But there are 201 integers.

By pigeonhole principle, there are must have $n_i\%100 = n_j\%100$, and there must be n_k s.t because we have $200 > 101$ numbers, $n_k\%100 = n_i\%100 = n_j\%100$ because we have 201 integers and there are 200 holes

i.e $\exists n_i, n_j, n_k \in \{n_1 \cdots n_{201}\}$

s.t $n_i\%100 = n_j\%100 = n_k\%100$

$\therefore 100 | (n_i - n_j)$ $100 | (n_j - n_k)$

$100 | (n_i - n_k)$