Midterm Solutions.

Q2 Pigeonhole.

1) Suppose Class A has 200 people. Assume we have a large box with green, yellow, and orange balls. Each student chooses 6 balls. Show there are at least 8 students who draw the same number of each color of ball.

Solution:

Step 1: How many different selection possibilities are there?

We have 3 colors (selections) and we are choosing 6 balls with possible repetition in color.

So, there are

$$C(6+3-1, 3-1) = C(8,2)$$

$$= \frac{8!}{6! \cdot 2!} = \frac{8 \cdot 7}{2} = 28$$

different selection possibilities.

Step 2:

Each person choose a selection,

We think of a students selection as a function from students to selection possibilities.

By the Pidgeonhole principle, we have that

$\left\lceil \frac{200}{28} \right\rceil = \lceil 7.1\ldots \rceil = 8$ students are mapped to the same selection.

This completes the proof. $\blacksquare$

2.) Assume there is another class, Class B.

Each student writes a word of length 6 using letters from $\{a, b, c\}$.

What is the minimum # of students the

class needs to have so that two students write the same word?

<u>Solution</u>!

How many possible words are there?

<u>Step 1</u>: write first letter $(n_1 = 3)$

⋮

<u>Step 6</u>: write 6^{th} letter $(n_6 = 3)$

By the multiplication principle, there are $3^6 = 729$ possible words.

We want two students to write the same word.

Consider a class with 730 students.

Let $f: \text{Class} \to \{\text{word}\}$ where

$f(\text{Student}) =$ word student has written.

By pigeonhole, there exists two students who have written the same word.

If the class has size less than or equal to 729, we could make each student write a distinct word.

So, $\boxed{730 \text{ Students.}}$

Q3) Let $\{S_n\}_{n=1}^{\infty}$ be a Sequence such that

1) $2S_n - 14S_{n-1} + 24S_{n-2} = 0$ for $n \geq 3$.

2) $S_1 = S_2 = 1$.

Find $\{S_n\}_{n=1}^{\infty}$.

Solution:

We have a homogenous recurrence relation of degree 2.

Notice;

$$2S_n - 14S_{n-1} + 24S_{n-2} = 0$$

$$\Leftrightarrow \quad S_n = 7S_{n-1} - 12S_{n-2}.$$

Let's solve this relation.

The associated polynomial equation is.

$$t^2 - 7t + 12 = 0$$

$$\Rightarrow (t-3)(t-4) = 0.$$

So, this polynomial has solutions when

$$t = 3 \quad \text{or} \quad t = 4.$$

So, $S_n = b4^n + d3^n$.

The initial conditions give two equations with two unknowns.

$$S_1 = 1 = b4 + d3$$

$$S_2 = 1 = b16 + d9$$

$1 = b4 + d3 \Rightarrow \dfrac{1-d3}{4} = b.$

So, $1 = b16 + d9 \Rightarrow 1 = \left(\dfrac{1-d3}{4}\right)16 + d9$

$= 4 - 12d + d9$

$= 4 - 3d$

$\Rightarrow -3 = -3d$

$\Rightarrow d = 1.$

Now

$1 = 4b + 3d \Rightarrow 1 = 4b + 3$

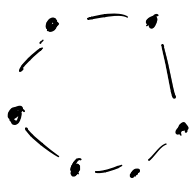
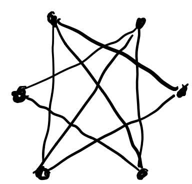
$\Rightarrow -2 = 4b$

$\Rightarrow b = -\frac{1}{2}.$

So, $\boxed{S_n = \left(-\frac{1}{2}\right)4^n + 3^n.}$

Q4)

1) Does there exist graph G on 6 vertices such that G, G' are connected?

Yes; $G =$  $G' =$ 

By observation, they are both connected.

2) Does there exist a graph G on 6 vertices such that G and G' both contain an Euler Cycle?

No; Assume toward a contradiction that the answer is yes.

Let $G = (V, E)$ and $v \in V$.

Since G contains an Euler cycle, the degree of v (in G) is even.

Now,

$$|V| = \Big| \{v\} \cup \{ w \in V = V' \mid \text{This is an edge from } w \text{ to } v \} \cup \{ w \in V = V' \mid \text{There is not an edge from } w \text{ to } v \text{ and } w \neq v \} \Big|$$

$$= 1 + \text{degree of } v \text{ in } G + \text{degree of } v \text{ in } G'$$

So $6 = 1 + \text{degree of } v \text{ in } G + \text{degree of } v \text{ in } G'$

So $\underbrace{\underbrace{5}_{\text{odd}} - \underbrace{\text{degree of } v \text{ in } G}_{\text{even}}}_{\text{odd.}} = \text{degree of } v \text{ in } G'$

So, degree of v in G' is odd,

So G' cannot have an Euler cycle.

Which is a contradiction.

∎

Q5 Let $X = \{1, \dots, 10\}$.

Let $V = \{A \mid A \subseteq X, |A| = 2\}$.

For any two distinct vertices $v, w \in V$, there is an edge between v and w if and only if $v \cap w = \emptyset$.

1) G is connected.

Let $v = \{a, b\}$, $w = \{d, c\}$ be in V.

We have two possibilities.

Case 1: $v \cap w \neq \emptyset$.

By construction, there is an edge from v to w.

So there is clearly a path $(v, e_{v,w}, w)$

Case 2: $v \cap w = \emptyset$.

Consider the element $t = \{a, c\}$.

Then, $v \cap t \neq \emptyset$, so we have an edge from v to t (say e_1).

Also, $w \cap t \neq \emptyset$, so we have an edge from t to w (say e_2).

So, $v e_1 t e_2 w$ is a path from v to w.

2) The degree $\{1,2\}$ is 16.

$\{1,2\}$ is connected to

$$\underbrace{\{1,3\} \text{---} \{1,10\}}_{8} \text{ and } \underbrace{\{2,3\}, \ldots, \{2,10\}}_{8}.$$

So, $\delta(\{1,2\}) = 8+8 = 16$.

3) G contains an Euler Cycle.

By Symmetry, every vertex has degree 16.

Since G is <u>Connected</u> and all vertices have even degrees G contains an Euler Cycle.