Q2 Induction

10 Points

Let $(a_n)_{n\in\mathbb{N}}$ the be Fibonacci sequence. Explicitly, $a_1 = a_2 = 1$ and for $n \geq 3$, we have $a_n = a_{n-1} + a_{n-2}$.

1. Show by induction that for every $n \in \mathbb{N}$, we have that

$$\sum_{i=1}^{n} a_i = a_{n+2} - 1.$$

2. Show by induction that for every $n \in \mathbb{N}$, we have that

$$\sum_{i=1}^{n} a_i^2 = a_n a_{n+1}.$$

Make sure to explicitly write out your base case and induction hypothesis.

Q3 Pigeon Hole

10 Points

Assume that there is another class, Class A, which has a 12 question, multiple choice final exam. The exam is broken into two sections:

a. The first section of the exam has 5 questions. Each of these questions has three possible answers (a,b, or c).

b. The second section of the exam has 7 questions. Each of these questions has four possible answers (a,b,c, or d).

Questions:

1. What is the minimum number of students which needs to be in Class A in order to guarantee that at least 3 students have the same exam solutions if we assume that no questions are left blank? [Every question is answered by every student].
2. What is the minimum number of students which needs to be in Class A in order to guarantee that at least 3 students have the same exam solutions if we accept "no solution" as a possible answer?

Q4 Recurrence Relation

10 Points

Let b_n be the number of paths in the graph $K_{3,3}$ of length exactly n.

Recall that the length of a path (in an unweighted simple graph) is the number of edges with appear in the path (they do not need to be distinct!). For instance, something like $P = (a, e_1, b, e_1, a, e_2, c)$ is a path of length 3. $P' = (c, e_2, a, e_1, b, e_1, a)$ is also a path of length 3 **distinct** from P (P and P' are counted separately). $P'' = (c, e_2, a, e_2, c, e_2, a)$ is also a path of length 3.

Since there are emails:

1. Write a recurrence relation for the number of paths on $K_{3,3}$. Explicitly, your equation should write b_n in terms of b_{n-1} (for $n \geq 2$). Justify your equation.
2. What is b_1? Justify your answer.
3. With your answers for part (1) and (2), solve the recurrence relation (your solution should be a formula which takes in n and returns b_n).
4. What is b_{100}? (Obviously, you do not need to "simply")

Q5 Adjacency Matrices

10 Points

Consider the following adjacency matrix:

$$A = \begin{pmatrix} & a & b & c & d \\ a & 0 & 1 & 1 & 1 \\ b & 1 & 0 & 0 & 1 \\ c & 1 & 0 & 0 & 1 \\ d & 1 & 1 & 1 & 0 \end{pmatrix}$$

Notice that

$$A^4 = \begin{pmatrix} & a & b & c & d \\ a & 15 & 9 & 9 & 14 \\ b & 9 & 10 & 10 & 9 \\ c & 9 & 10 & 10 & 9 \\ d & 14 & 9 & 9 & 15 \end{pmatrix}$$

1. Draw the graph associated to the adjacency matrix A (with the correct labeling).
2. Using the data provided, what is the total number of paths of length 4 in the graph associated to the adjacency matrix A? [Note: As in question 4, the path (a,b,c,d) is **different** than the path (d,c,b,a)].
3. Using the data provided, what is the number of paths of length 4 which start at a and **do not** end at a? [These paths can end at any other vertex, just not a].
4. **Using only the data provided above**, compute the number of paths of length 5 (reread: it says 5) from a to a. **You must show your scratch work**.

Q6 Interpretation and graphs #1

10 Points

(Part of this question is interpreting what is written.)

Let $X = \{1, 2, 3\}$. We define the following simple graph $G_1 = (V_1, E_1)$ as follows

1. $V_1 = \{A : A \subseteq X, |A| > 0\}$.
2. Let v and w be two distinct elements of V_1. There is an edge between v and w if and only if $v \cap w \neq \emptyset$.

Questions:

1. Draw a picture of G_1 with vertices clearly labeled.
2. Does G_1 contain an Euler cycle? Prove your claim.
3. Is G_1 planar? Prove your claim.

Q7 Interpretation and graphs #2

10 Points

(Part of this question is interpreting what is written.)

Let $X = \{1, 2, 3\}$. We define the following simple graph $G_2 = (V_2, E_2)$ as follows:

1. $V_2 = X \times X$.
2. Let $v = (a, b)$ and $w = (c, d)$ be two distinct elements of V_2. There is an edge between v and w if and only if either $a = c + 1$ or $c = a + 1$.

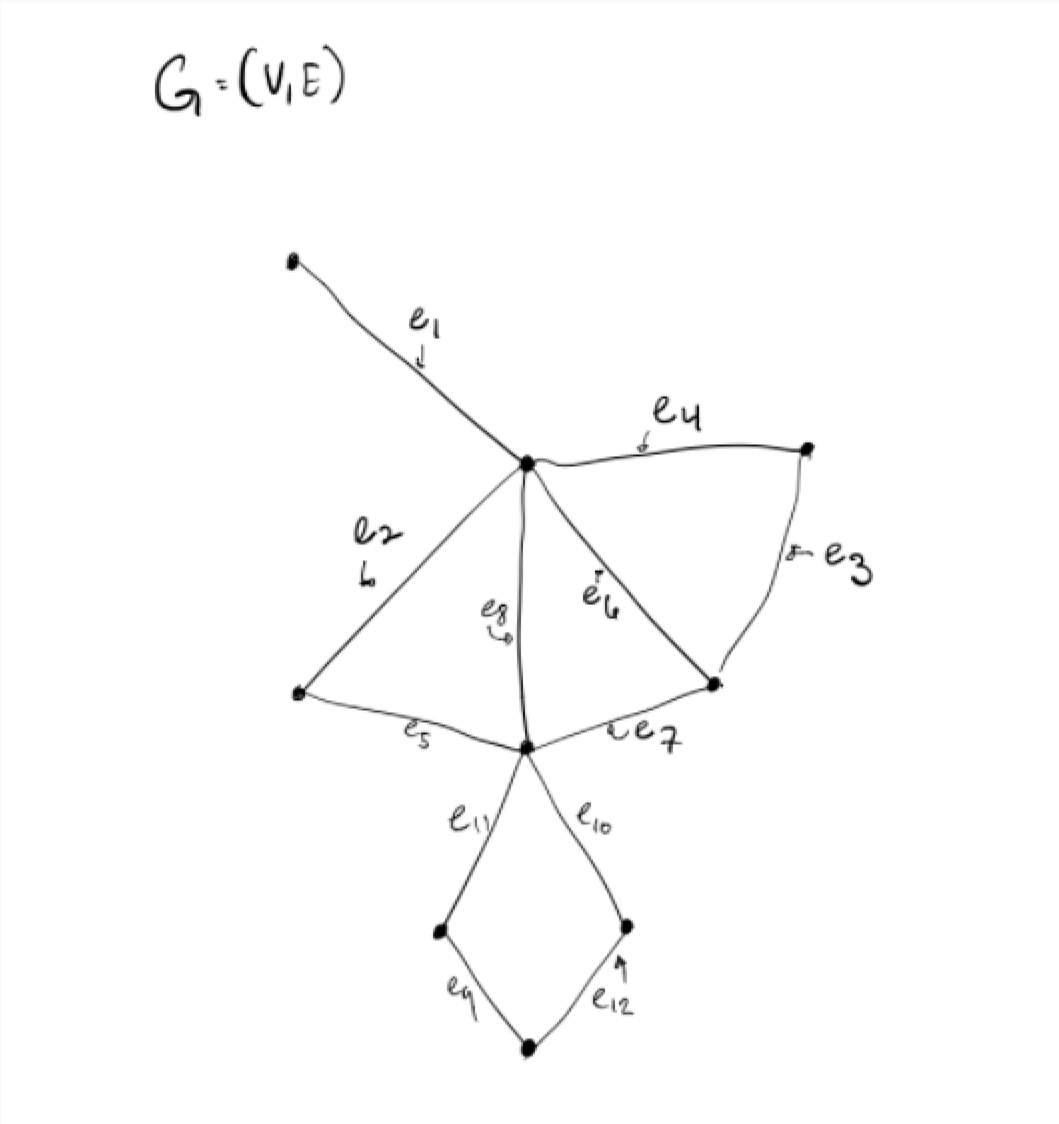
Questions:

1. Draw a picture of G_2 with vertices clearly labeled.
2. Does G_2 contain an Euler cycle? Prove your claim.
3. Is G_2 planar? Prove your claim.

Q8 Kruskal's Algorithm

10 Points

Consider the following graph $G = (V, E)$:



We now turn this graph into a weighted graph with the function $w : E \to \mathbb{R}$ where $w(e_i) = i$. So the edge e_1 has weight 1, the edge e_2 has weight 2,... and so on.

1. Use Kruskal's algorithm (the algorithm described in class) to find a minimal spanning tree. Be sure to show your work using the algorithm.
2. Draw the minimal spanning tree (with edges clearly labeled).
3. What is the weight of the minimal spanning tree?
4. Give an example of a different spanning tree of G. What is the weight of the tree you have given?

Q9 Examples

10 Points

The following four questions ask whether a simple graph satisfying a certain property exists. If yes, provide an example. If no, justify your answer with a brief proof.

1. Does there exist a tree $T = (V, E)$ such that $|V| \geq 1$ and $2|V| = \sum_{v \in V} \delta(v)$?
2. Does there exist a weighted simple graph $G = (V, E)$ (with weighting w) such that $|V| = 4$ and G has at least two distinct minimal spanning trees? (If yes, you must give a graph and a weight function on edges).
3. Does there exist a simple graph $G = (V, E)$ with 7 vertices and degrees $(2, 2, 2, 2, 3, 3, 3)$?
4. Does there exist a simple graph $G = (V, E)$ such that G is not connected and $|E| > |V| > 1$?

TOTAL POINTS

78 / 81 pts

QUESTION 1	
Honor Code	**1** / 1 pt
QUESTION 2	
Induction	**10** / 10 pts
QUESTION 3	
Pigeon Hole	**10** / 10 pts
QUESTION 4	
Recurrence Relation	**9** / 10 pts
QUESTION 5	
Adjacency Matrices	**10** / 10 pts
QUESTION 6	
Interpretation and graphs #1	**10** / 10 pts
QUESTION 7	
Interpretation and graphs #2	**10** / 10 pts
QUESTION 8	
Kruskal's Algorithm	**10** / 10 pts
QUESTION 9	
Examples	**8** / 10 pts

QUESTION 4

Recurrence Relation **9** / 10 pts

Part 1

	+ 3 pts	Correct
✔	**+ 2 pts**	Correct answer but insufficient justification, e.g. claiming without proof a property of the adjacency matrix powers
	+ 2 pts	Correct recurrence of the wrong order
	+ 2 pts	Minor error in reasoning
	+ 1 pt	Has the right idea to build a recurrence by extending a path via adding a new edge, but does not correctly combine "number of paths of length $n-1$" with "number of ways to add one more edge"
	+ 1 pt	Attempts to use powers of the adjacency matrix A but writes down the wrong A
	+ 0 pts	Incorrect

QUESTION 9

Examples

8 / 10 pts

(1)

+ 3 pts (1): Correct

✔ **+ 2.5 pts** (1): Subtly/slightly incomplete/incorrect argument

+ 2 pts (1): Incomplete argument

+ 1.5 pts (1): Partially correct

+ 1 pt (1): Unclear and incomplete

+ 0.5 pts (1): Argument makes no real progress

+ 0 pts (1): Example is not a tree

+ 0 pts (1): Incorrect/missing

(2)

✔ **+ 3 pts** (2): Correct

+ 2.5 pts (2): Correct weighted graph, no example MSTs

+ 2.5 pts (2): Weighted graph not given, intention clear from MSTs

+ 2 pts (2): Insufficiently/incorrectly specified weight function

+ 2 pts (2): Correct weighted graph, give non-spanning trees

+ 1.5 pts (2): Partially correct argument, no examples

+ 1.5 pts (2): Correct approach, given spanning trees are not minimal

+ 1 pt (2): Incorrect approach, given spanning trees are not minimal

+ 1 pt (2): No weights

+ 0 pts (2): Incorrect/missing

(3)

+ 2 pts (3): Correct

+ 1.5 pts (3): Correct reason, incorrect justification

+ 1 pt (3): Partial argument

✔ **+ 0.5 pts** (3): Very partial argument

+ 0 pts (3): Incorrect/missing

(4)

✔ **+ 2 pts** (4): Correct

(1): Your proof only works for trees constructed one edge and one vertex at a time through a series of intermediate trees. You are not given such a construction, however; if you want to use it, you need to prove that it exists.

(3): You are assuming that there are no edges between the vertices of degree 3. This is not justified.

$a_1 = a_2 = 1$
for $n \geq 3$, we have
$a_n = a_{n-1} + a_{n-2}$

Math 61 - Final (Question 2) (page 1)

(1) proof: Let $P(n) := \text{"} \sum_{i=1}^{n} a_i = a_{n+2} - 1 \text{"}$

base case: WTS that $P(1)$ holds.

$P(1) := \text{"} \sum_{i=1}^{1} a_i = a_{1+2} - 1 \text{"}$

Notice that $\sum_{i=1}^{1} a_i = a_1 = 1$

Notice that $a_{1+2} - 1 = a_3 - 1 = (a_2 + a_1) - 1 = 2 - 1 = 1$

Since $1 = 1$, $P(1)$ holds. ✓

induction hypothesis: Assume that $P(n)$ holds.

induction step: WTS $P(n+1)$ is true.

Notice that $P(n+1) := \text{"} \sum_{i=1}^{n+1} a_i = a_{(n+1)+2} - 1 \text{"}$

$$\sum_{i=1}^{n+1} a_i = \left(\sum_{i=1}^{n} a_i\right) + a_{n+1}$$

(by I.H.) $\longrightarrow$ $= (a_{n+2} - 1) + a_{n+1}$

$= a_{n+2} + a_{n+1} - 1$

* Recall that $a_n = a_{n-1} + a_{n-2}$

Notice that

$$a_{n+2} + a_{n+1} - 1 = a_{(n+3)-1} + a_{(n+3)-2} - 1$$
$$= a_{n+3} - 1 \checkmark$$

Hence, if $P(n)$ is true, then $P(n+1)$ is true. So by the principle of mathematic induction, we conclude that for every $n \in \mathbb{N}$, we have that $\sum_{i=1}^{n} a_i = a_{n+2} - 1$.

$\blacksquare$

$a_1 = a_2 = 1$
for $n \geq 3$, we have
$a_n = a_{n-1} + a_{n-2}$

Math 61 - Final (Question 2) (page 2)

(2) proof: Let $P(n) := $ "$\sum_{i=1}^{n} a_i^2 = a_n a_{n+1}$"

base case: WTS that $P(1)$ holds.

$P(1) := $ "$\sum_{i=1}^{1} a_i^2 = a_1 a_{1+1}$"

Notice that $\sum_{i=1}^{1} a_i^2 = (a_1)^2 = 1^2 = 1$

Notice that $a_1 a_{1+1} = a_1 \cdot a_2 = 1 \cdot 1 = 1$

Since $1 = 1$, $P(1)$ holds. ✓

inductive hypothesis: Assume $P(n)$ holds.

inductive step: WTS $P(n+1)$ is true.

Notice that $P(n+1) := $ "$\sum_{i=1}^{n+1} a_i^2 = (a_{n+1})(a_{n+1+1})$"

$$\sum_{i=1}^{n+1} a_i^2 = \left(\sum_{i=1}^{n} a_i^2\right) + (a_{n+1})^2$$

(by I.H.) $\longrightarrow$ $$= (a_n a_{n+1}) + (a_{n+1})^2$$
$$= (a_{n+1})(a_n + a_{n+1})$$

* Recall that $a_n = a_{n-1} + a_{n-2}$

Notice that

$$(a_{n+1})(a_n + a_{n+1}) = (a_{n+1})(a_{(n+2)-2} + a_{(n+2)-1})$$
$$= (a_{n+1})(a_{n+2}) \checkmark$$

Hence, if $P(n)$ is true, then $P(n+1)$ is true. So by the principle of mathematical induction, we conclude that for every $n \in \mathbb{N}$, we have that $\sum_{i=1}^{n} a_i^2 = a_n a_{n+1}$.

∎

Math 61 - Final (Question 3) (part 1)

(1) Step 1: How many unique solutions are possible?

calculation:

Step 1: Choose an answer for Q1 ($n_1 = 3$)
⋮
Step 5: Choose an answer for Q5 ($n_5 = 3$)
} first section

Step 6: Choose an answer for Q6 ($n_6 = 4$)
⋮
Step 12: Choose an answer for Q12 ($n_{12} = 4$)
} second section

By multiplication principle, there are $3^5 \cdot 4^7 = 3981312$ unique solutions

Step 2: find min. # of students to guarantee at least 3 students have the same exam solutions via pigeonhole principle.

- there are 3981312 exam solutions
  - we have 3981312 pigeonholes (n)
- want to find # of pigeons (students) s.t. at least 3 share the same pigeonhole ; # of pigeons = k
- need $k > n$ for pidgeonhole principle to apply
- if looking for min. # of students so at least 2 share the same exam solution
  - min # of students = $n+1$
- so what about so 3 students share the same exam solution
  - min # of students = $2n+1 = \boxed{7{,}962{,}625 \text{ students}}$

Check:

- $n < k \Rightarrow 3981312 < 7962625$ ✓
- $\lceil \frac{k}{n} \rceil = \lceil \frac{7962625}{3981312} \rceil = 3$ ✓
- Since $n < k$, by Pidgeonhole principle, there are at least $\lceil \frac{k}{n} \rceil = 3$ students with the same exam solutions.

Math 61 - Final (Question 3) (part 2)

(2) Let's treat "no solution" as another option for a question.
Thus, we follow the same steps as (1) but with each question having +1 choices.

Step 1: How many unique solutions are possible?

calculation:

Step 1: Choose an answer for Q1 $(n_1 = 4)$
⋮
Step 5: Choose an answer for Q5 $(n_5 = 4)$

$\}$ first section

Step 6: Choose an answer for Q6 $(n_6 = 5)$
⋮
Step 12: Choose an answer for Q12 $(n_{12} = 5)$

$\}$ second section

By multiplication principle, there are $4^5 \cdot 5^7 = 80{,}000{,}000$ unique solutions

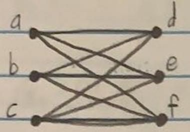
Step 2: find min. # of students to guarantee at least 3 students have the the same exam solutions via pigeonhole principle.

- there are 80,000,000 exam solutions
  - we have 80,000,000 pigeonholes
- want to find # of pigeons (students) s.t. at least 3 share the same pigeonhole ; # of pigeons $= k$
- need $k > n$ for pidgeonhole principle to apply
- if looking for min # of students so at least 2 share the same exam solution
  - min # of students $= n+1$
- so what about so 3 students share the same exam solution
  - min # of students $= 2n+1 = \boxed{160{,}000{,}001 \text{ students}}$

Check:

- $n < k \Rightarrow 80{,}000{,}000 < 160{,}000{,}001$ ✓
- $\left\lceil \frac{k}{n} \right\rceil = \left\lceil \frac{160000001}{80000000} \right\rceil = 3$
- Since $n < k$, by Pidgeonhole principle, there are at least $\left\lceil \frac{k}{n} \right\rceil = 3$ students with the same exam solutions.

Math 61 - Final (Question 4) (page 1)

(1) 

$= K_{3,3}$

(* the ij^{th} entry in the matrix M^n is equal to the number of paths from i to j of length n.)

$$b_1 = M = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

length 1: 18 paths

Notice that
$$M^2 = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{pmatrix} 3 & 3 & 3 & 0 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 3 & 3 & 3 \end{pmatrix} \end{array}$$
$(M^2 = b_2)$

length 2: 54 paths

Notice that
$$M^3 = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 9 & 9 & 9 \\ 0 & 0 & 0 & 9 & 9 & 9 \\ 0 & 0 & 0 & 9 & 9 & 9 \\ 9 & 9 & 9 & 0 & 0 & 0 \\ 9 & 9 & 9 & 0 & 0 & 0 \\ 9 & 9 & 9 & 0 & 0 & 0 \end{pmatrix} \end{array}$$
$(M^3 = b_3)$

length 3: 162 paths

Growth by factor of 3 between b_n and b_{n+1} where n = length of path

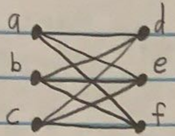
Thus,

$$b_n = 3 \cdot b_{n-1} \quad \text{for } n \geq 2$$

Math 61 - Final (Question 4) (page 2)

(2) $b_1 = 18$

Notice that $K_{3,3}$ can be represented as a matrix.



$$M' = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Notice that the ij^{th} entry in the matrix M^n is equal to the number of paths from i to j of length n.

Notice the path a to j and path j to a are counted separately.

Thus adding all values in M will give b_1 since it is the number of paths in graph $K_{3,3}$ of length exactly n.

Doing so, $b_1 = 18$.

Note: the same answer can be reached by counting the paths of length 1 by looking at graph $K_{3,3}$. It is more tedious though.

(3) $b_n = 3 \cdot b_{n-1}$; initial condition $b_1 = 18$

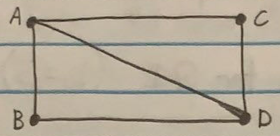
solution:
$$\begin{aligned} b_n &= 3b_{n-1} \\ &= 3(3b_{n-2}) = 3^2 b_{n-2} \\ &= 3^2(3b_{n-3}) = 3^3 b_{n-3} \\ &= 3^3(3b_{n-4}) = 3^4 b_{n-4} \\ &\vdots \\ &= 3^{n-1} b_1 \\ &= 3^{n-1} \cdot 18 \\ &= 3^n \cdot \tfrac{1}{3} \cdot 18 \end{aligned}$$

$$\boxed{b_n = 3^n \cdot 6}$$

(4)
$$\begin{aligned} b_{100} &= 3^{100} \cdot 6 \\ &= 3.092265 \cdot 10^{48} \end{aligned}$$

Math 61 - Final (Question 5)

(1)



(2) $(9+9+14+10+9+9)\cdot 2 + (15+10+10+15)$

$= (60)\cdot 2 + (50)$

$= \boxed{170}$

[a,a] [a,b] [a,c] [a,d]
[b,a] [b,b] [b,c] [b,d]
[c,a] [c,b] [c,c] [c,d]
[d,a] [d,b] [d,c] [d,d]

(3) $9+9+14 = 18+14 = \boxed{32}$

(4) $$\begin{pmatrix} 15 & 9 & 9 & 14 \\ 9 & 10 & 10 & 9 \\ 9 & 10 & 10 & 9 \\ 14 & 9 & 9 & 15 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 32 & & & \\ & & & \\ & & & \\ & & & \end{pmatrix}$$

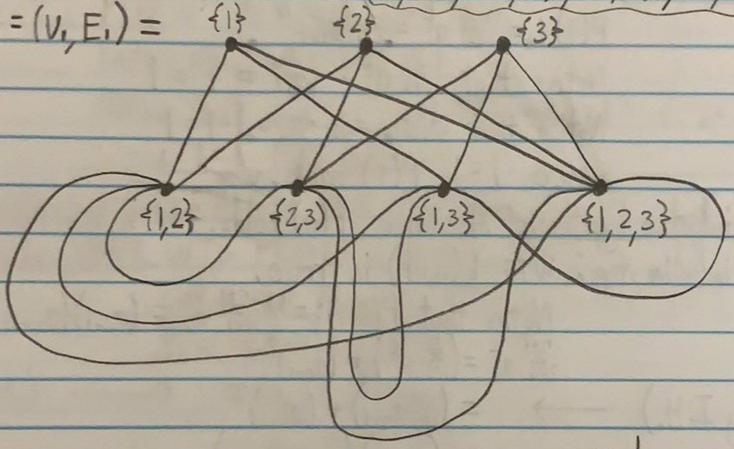
* ij^{th} entry in A^{n+1} is gotten by multiplying i^{th} row in A^n by the j^{th} column in A.

$$\begin{pmatrix} 15 & 9 & 9 & 14 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} = 15\cdot 0 + 9\cdot 1 + 9\cdot 1 + 14\cdot 1 = 9+9+14 = \boxed{32}$$

Math 61 - Final (Question 6)

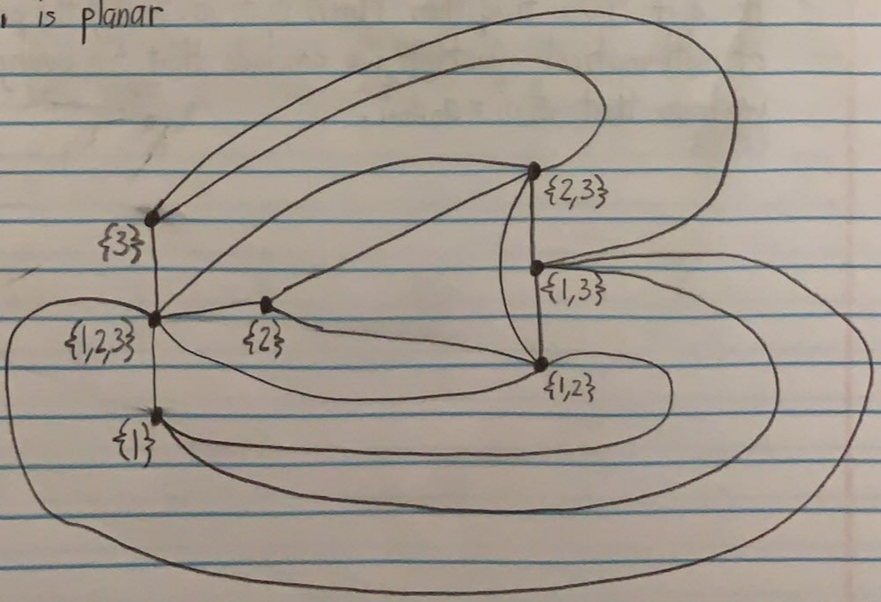
(1) $V_1 = \{A : A \subseteq X, |A| > 0\} = \{1\}, \{2\}, \{3\},$
$\{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}$

$G_1 = (V_1, E_1) =$



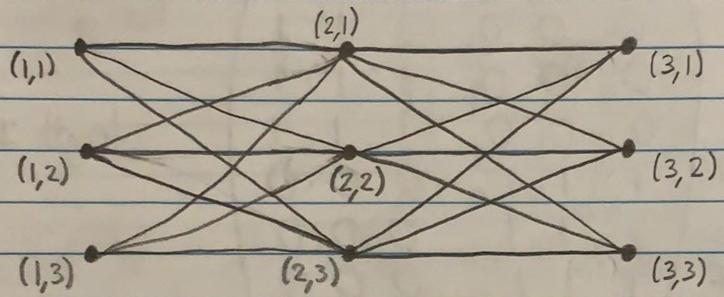
(2) A graph has an Euler cycle if it is connected and all vertices have even degree.
Notice that $\delta(\{1\}) = 3$ (AKA odd).
Since not every vertex in G_1 has even degree, G' does NOT contain an Euler cycle.

(3) G_1 is planar



Math 61 - Final (Question 7)

(1) $V_2 = X \times X = \left\{ \begin{matrix} (1,1), & (1,2), & (1,3), \\ (2,1), & (2,2), & (2,3), \\ (3,1), & (3,2), & (3,3) \end{matrix} \right\}$



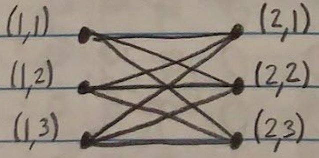
(2) A graph has an Euler cycle if it is connected and all vertices have even degree.

Notice that $\delta((1,1)) = 3$

Since not every vertex in G_2 has even degree, G_2 does NOT contain an Euler cycle.

(3) G_2 is NOT planar since it contains a subgraph homeomorphic to $K_{3,3}$.

Notice G_2 contains subgraph $K_{3,3}$.

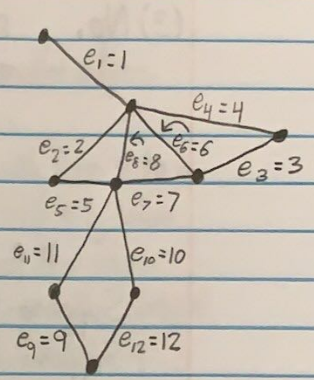


A graph G is planar if and only if G does not contain a subgraph homeomorphic to K_5 or $K_{3,3}$.

Since G_2 contains subgraph $K_{3,3}$, it is NOT planar.

Math 61 - Final (Question 8) (page 1)

(1) $G = (V, E)$; weight function $w : E \to \mathbb{R}$ where $w(e_i) = i$



Step 0:

$E = e_1, e_2, \ldots, e_{12}$

Step 1: Let $T_0 = (V, E_0)$ where $E_0 = \emptyset$.

$T_1 = T_0 \cup \{e_1\}$

Step 2: We have T_1.

Consider edge e_2. Does not make a cycle.

So, $T_2 = T_1 \cup \{e_2\}$.

Step 3: We have T_2.

Consider edge e_3. Does not make a cycle.

So, $T_3 = T_2 \cup \{e_3\}$.

Step 4: We have T_3.

Consider e_4. Does not make a cycle.

So, $T_4 = T_3 \cup \{e_4\}$.

Step 5: We have T_4. Consider e_5.

Does not make a cycle. So, $T_5 = T_4 \cup \{e_5\}$.

Step 6: We have T_5. Consider e_6.

Makes a cycle. So, $T_6 = T_5$.

Step 7: We have T_6. Consider e_7.

Makes a cycle. So, $T_7 = T_6$.

Step 8: We have T_7. Consider e_8.

Makes a cycle. So, $T_8 = T_7$.

Step 9: We have T_8. Consider e_9.

Does not make a cycle. So, $T_9 = T_8 \cup \{e_9\}$.

Step 10: We have T_9. Consider e_{10}.

Does not make a cycle. So, $T_{10} = T_9 \cup \{e_{10}\}$.

Step 11: We have T_{10}. Consider e_{11}.

Does not make a cycle. So, $T_{11} = T_{10} \cup \{e_{11}\}$.

Step 12: We have T_{11}. Consider e_{12}.

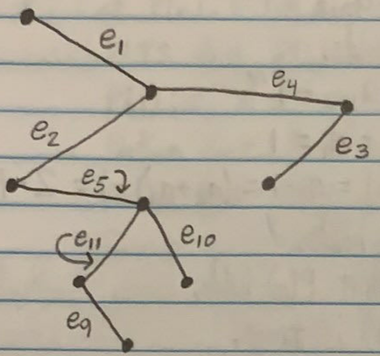
Makes a cycle. So, $T_{12} = T_{11}$.

Return T_{12}.

$T_{12} = (V, E_{12})$ where $E_{12} = \{e_1, e_2, e_3, e_4, e_5, e_9, e_{10}, e_{11}\}$

Math 61 - Final (Question 8) (page 2)

(2)

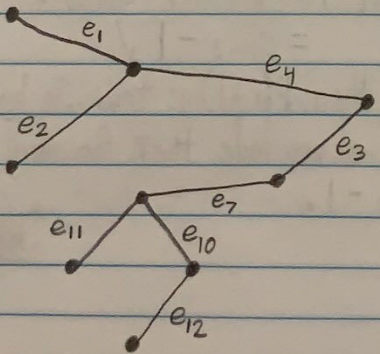


(3) $G=(V,E)$ w/ weight function $w: E \to \mathbb{R}$ where $w(e_i)=i$.

So,

$$\text{weight} = w(e_1)+w(e_2)+w(e_3)+w(e_4)+w(e_5)+w(e_9)+w(e_{10})+w(e_{11})$$
$$= 1+2+3+4+5+9+10+11$$
$$= 15+20+10$$
$$= \boxed{45}$$

(4)



$$\text{weight} = w(e_1)+w(e_2)+w(e_3)+w(e_4)+w(e_7)+w(e_{10})+w(e_{11})+w(e_{12})$$
$$= 1+2+3+4+7+10+11+12$$
$$= 10+17+23$$
$$= \boxed{50}$$

Math 61 - Final (Question 9) (page 1)

(1) No.

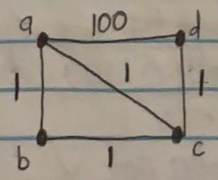
A tree is a simple graph, thereby having no loops or parallel edges.
A tree also has no cycles.
If v, w are vertices in T, then there is a unique simple path from v to w.
With a tree being held by such restrictions, for $|V| \geq 1$, when an edge is added, a new vertex must be at one end of the edge.

$|V| = 1$; $2|V| = 2$, $\sum_{v \in V} \delta(v) = 0$ $\quad$ $|V| = 2$, $2|V| = 4$, $\sum_{v \in V} \delta(v) = 2$ $\quad$ $|V| = 3$, $2|V| = 6$, $\sum_{v \in V} \delta(v) = 4$

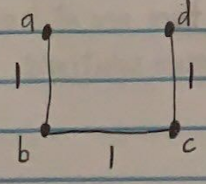
Notice that for every vertex added, total degree of the tree always increases by 2 if $|V| > 1$.
This is because when a new edge is added, it adds 1 degree to the vertex it connects to on one end, and adds 1 degree to the new vertex it connects to on the other end.
Notice though that when the first node of a tree is added, degree count does not increase.
Since the addition of each vertex after the very first vertex is added increases degree count by 2, no more, no less, for $|V| \geq 1$ $2|V|$ will always by greater than $\sum_{v \in V} \delta(v)$ by 2.
There is no way to recovery the +2 that $2|V|$ has over $\sum_{v \in V} \delta(v)$, so there does not exist a tree $T = (V, E)$ s.t. $|V| \geq 1$ and $2|V| = \sum_{v \in V} \delta(v)$.

(2) Yes.

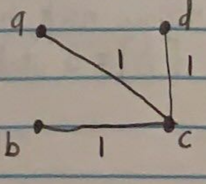
G =



minimal spanning trees:



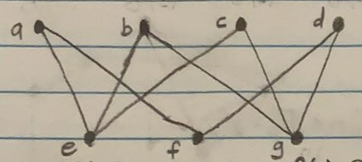
weight: 3



weight: 3

Math 61 - Final (Question 9) (page 2)

(3) No. $\delta(a)=2$ $\delta(b)=2$ $\delta(c)=2$ $\delta(d)=2$



$\delta(e)=3$ $\delta(f)=2$ $\delta(g)=3$

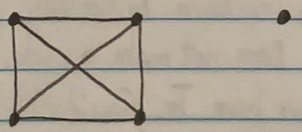
messing up property

* f needs one more degree, but no place to make it happen without

Suppose that there is a simple graph w/ vertices a, b, c, d, e, f, g. Suppose that the degrees of e, f, g are 3. Since the graph is simple, no loops or parallel edges are allowed and thus the degrees of a, b, c, d are at least 2 but one of the 4 vertices must have degree 3. Thus, there is no such graph fitting the given problem description.

(4) Yes.

Let $G=(V,E)$ be



$|V|=5$

$|E|=6$

Thus, $|E|>|V|>1$.

G is not connected since there does not exist a path between any two vertices in G.