DO NOT OPEN THIS TEST UNTIL YOU ARE INSTRUCTED TO DO SO!

Class: Math 61, Lecture 1
Instructor: Jonathan Rubin
Exam: Midterm I
Date: 21 October 2019
Time: 11:00 AM – 11:50 AM

THIS IS A CLOSED BOOK EXAM. NO OUTSIDE AIDS, SUCH AS NOTES, TEXTBOOKS, CALCULATORS, OR CELLPHONES ARE PERMITTED.

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*I understand that this is a closed book exam. I certify that the following work is mine alone, and I pledge that I have neither given nor received unauthorized assistance on this test.*

Signature: David Morley

**Instructions:** This is a 50-minute exam. It consists of four problems, and there is an extra piece of scratch paper at the end. Please write your answers in the space provided. If you run out of room, then please continue onto the back of the page and indicate clearly that you have done so. **Good luck!**

| Question | Points | Score |
|---|---|---|
| 1 | 10 | |
| 2 | 10 | |
| 3 | 10 | |
| 4 | 10 | |
| Total: | 40 | |

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1. (10 points) How many 5-letter (possibly nonsense) words $w$ are there in $\{A, B, C, \ldots, Z\}$ that satisfy the following four properties?

   (i) The word $w$ starts with an $A$ or $B$, and

   (ii) the word $w$ ends with an $X$, $Y$, or $Z$, and

   (iii) the word $w$ contains at least one $C$, and

   (iv) the word $w$ contains exactly one $D$.

   Please simplify your answer as much as possible.

X Computed each part as if future conditions were not applied, then applied them

i) Starts with an a or b (if no other restrictions)

By the multiplication principle $2 \cdot 26P4$

ii) ends with an x y or z

$2 \cdot 26P3 \cdot 3$

iii) + iv) _ _ _ _ _ (2 options, 3 options) contains either one C or 2 C's

word contains either 1 C or 2 C's

There are two possible cases with the C and D restrictions
1 There is 1 C, 1 D and 1 other letter
2 There are 2 Cs and 1 D ~~26C1~~

$\boxed{1C\ case}$ $2 \cdot 3 \cdot 3C1 \cdot 2C1 \cdot 24$ — remaining possible letters

start and end; select spot for C; select spot for D

$\boxed{2C\ case}$ for the 2 C case

$2 \cdot 3 \cdot 3C2 \cdot 1$ — only one option for the D

start and end; choose 2 C's

total words

By the Addition Principle

$6 \cdot [6(24) + 6]$ ~~36·24~~ ~~= (36)^2 24~~ $= \boxed{36(25)}$ answer $36[24+1]$ $\approx 900$ words

3

2. (10 points) Consider the binary relation $R$ on $\{1, 2, 3, 4\}$ whose matrix is

T
$$\begin{array}{c|c} & \begin{matrix}1 & 2 & 3 & 4\end{matrix} \\ \begin{matrix}1\\2\\3\\4\end{matrix} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}.$$

This means that $iRj$ is true if the $(i, j)$-entry is 1, and false if it is 0.

(a) (2 points) Is the relation $R$ symmetric? Briefly explain.

The relation is $\boxed{\text{not}}$ symmetric as reflecting it over this diagonal does not result in the same matrix.

def of symmetric: if ARB then BRA for (A, B) ∈ X as 2 R 1 holds but 1 R 2 doesn't (column) (across) not symmetric

(b) (2 points) Is the relation $R$ reflexive? Briefly explain.

Definition: if a ∈ X then aRa for all values of a in X

$\boxed{\text{"No"}}$ In order for a matrix to be reflexive (1, 1) (2, 2) (3, 3), ..., must be included in the relation as (1, 1) is not in the relation ∴ not reflexive.

(c) (3 points) Is the relation $R$ transitive? Briefly explain.

Transitive: By definition if R∘R reflects R then transitive (change >1 to 1)

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 1 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$ This matrix simplifies to the initial

$\boxed{\therefore \text{ transitive}}$

$\boxed{\text{"Yes" it is transitive}}$

(d) (3 points) Is the relation $R$ antisymmetric? Briefly explain.

antisymmetric if ARB then BRA false for (A, B) ∈ X

this $\boxed{\text{is}}$ antisymmetric as flipping it over this diagonal shows that all 1's and 0's are paired oppositely

↳ this means that ARB ≠ BRA, therefore antisymmetricity holds

According to TA is what syntax

$a \to \{a\}$ $d \to \{d\}$ means

$b \to \{b\}$

$c \to \{c\}$

take an element

$\{x\} \to$ map

3. (10 points) Consider the function

$$s(x) = \{x\} : \{a, b, c, d\} \to \mathcal{P}(\{a, b, c, d\}),$$

where $\mathcal{P}(\{a, b, c, d\})$ is the set of all subsets of $\{a, b, c, d\}$. This means that the function $s$ sends a letter $x \in \{a, b, c, d\}$ to the *singleton set* $\{x\}$, whose only element is $x$.

(a) (5 points) Is the function $s$ surjective? Justify your answer.

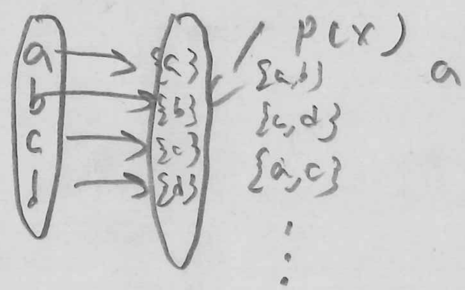
Surjective: definition → for an element $y \in Y$ there exists an $x \in X$ such that $\boxed{f(x) = y}$

$P(\{a, b, c, d\}$ → set of all subsets

Draw a picture

takes the element

$x \to$ maps it to the set $x$ which is in the power set



As the problem states that one value of $x$ either $a, b, c$, or $d$ is mapped to its equivalent set either $\{a\}, \{b\}, \{c\}, \{d\}$ it can be proven that the function is $\boxed{\text{not}}$ surjective as the other elements in the codomain of $P\{a, b, c, d\}$ are not all mapped to by values of $x$, for example $\{a, b\}, \{c, d\}$ and many other sets are not included.

(b) (5 points) Is the function $s$ injective? Justify your answer.

Injective: definition for any $x_1, x_2 \in X$ if $f(x_1) = f(x_2)$ then $\boxed{x_1 = x_2}$ (one to one)

There are four possible values of $x$ → $a, b, c$ or $d$.

if $x = a \longrightarrow \{a\}$

if $x = b \longrightarrow \{b\}$

if $x = c \longrightarrow \{c\}$

if $x = d \longrightarrow \{d\}$

As all possible values in $X$ map to 1 distinct value in the codomain, and thus $S(x_1) = S(x_2)$ only if $x_1 = x_2$ this function is by definition $\boxed{\text{injective.}}$

Base case $n=1$ $\frac{1}{2} = \frac{1}{1+1}$

$\frac{1}{2} = \frac{1}{2}$ ✓ holds

4. (10 points) Prove that the equation

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \cdots + \frac{1}{n \cdot (n+1)} = \frac{n}{n+1}$$

holds for every integer $n \geq 1$.

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}.$$

Assume the induction n

$$\underbrace{\frac{1}{2} + \frac{1}{6} + \frac{1}{12} \cdots}\ + \frac{1}{(n+1)(n+2)} = \frac{(n+1)}{n+2}$$

Now substitute

Find a common denominator

$$\frac{n}{n+1} + \frac{1}{(n+1)(n+2)} = \frac{n+1}{n+2}$$

multiply out

$$\frac{n(n+2)+1}{(n+1)(n+2)} = \frac{n+1}{n+2}$$

Factor

$$\frac{n^2+2n+1}{(n+1)(n+2} = \frac{n+1}{n+2}$$

as $n \neq -1$ can cancel out $\frac{n+1}{n+1}$ $(n \geq 0)$

$$\frac{(n+1)^{\cancel{2}}}{\cancel{(n+1)}(n+2)} = \frac{n+1}{n+2}$$

Thus the left side equals the right side.

$$\frac{n+1}{n+2} = \frac{n+1}{n+2}$$ $\boxed{\phantom{x}}$ By induction

as $\frac{n+1}{n+2} = \frac{n+1}{n+2}$ this equation holds by induction

Extra scratch paper.

$\underline{\ 2\ }\ \_\ \_\ \_\ \underline{\ 3\ }$

2 cases 1 C case

For 1C Case

I

$3C1 \cdot 2C1 \quad 4C1 \cdot 24$

options for C

options for d

the remaining letters that can be used (not C or D)

II 2 C case

$3C2 \cdot 1C1$ · the remaining letters

one option for d

$6 \cdot \left[ 6 \cdot 24 + 6 \cdot \right]$

3.1

$$\begin{array}{r} 36 \\ \underline{{}_1 25} \\ 180 \\ 720 \\ \hline 900 \end{array}$$

900

2 3

6 •

3 • 2 • 24

6 •

6 • [6 (24) + 6]

36 (25)