ANSWER KEY
MATH 61, LECTURE 1
MIDTERM I

1. Induction

Prove that the equation

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{2} + ... + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

holds for every integer $n \geq 1$.

Proof. Let P_n be the statement $\frac{1}{2} + \frac{1}{6} + \frac{1}{2} + ... + \frac{1}{n(n+1)} = \frac{n}{n+1}$. We proceed by induction on n.

First we show the base case $n = 1$. This says:

$$\frac{1}{1(1+1)} = \frac{1}{1+1},$$

which means P_1 is true.

Now let $n \geq 1$ be given, and suppose that P_n is true. This means that

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{2} + ... + \frac{1}{n(n+1)} = \frac{n}{n+1}.$$

Adding $\frac{1}{(n+1)(n+2)}$ to both sides of the equation, we obtain

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{2} + ... + \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} = \frac{n}{n+1} + \frac{1}{(n+1)(n+2)}.$$

Now we simplify the right hand side.

$$\begin{aligned}
\frac{n}{n+1} + \frac{1}{(n+1)(n+2)} &= \frac{n}{n+1} \cdot \frac{n+2}{n+2} + \frac{1}{(n+1)(n+2)} \\
&= \frac{n(n+2)}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)} \\
&= \frac{n^2+2n+1}{(n+1)(n+2)} \\
&= \frac{(n+1)^2}{(n+1)(n+2)} \\
&= \frac{n+1}{n+2}.
\end{aligned}$$

Hence $\frac{1}{2} + \frac{1}{6} + \frac{1}{2} + ... + \frac{1}{(n+1)(n+2)} = \frac{n+1}{n+2}$, which means P_{n+1} is true. We have shown that for all $n \geq 1$, P_n implies P_{n+1}.

By mathematical induction, P_n holds for all $n \geq 1$. □

2. Counting

How many five-letter words satisfy the following properties:

(1) The first letter is A or B.
(2) The last letter is X, Y, or Z.
(3) There is at least one C.
(4) There is exactly one D.

We have two choices for the first letter, three for the last letter, and three for the position of D (it can go into the second, third, or fourth position). After we have chosen these, we need to fill the remaining two spots with two letters, neither of which is a D and at least one of which is a C.

There are 25 pairs of letters where the first is a C and the second is not D, and there are 25 pairs of letters where the second is a C and the first is not D. Since there is one pair (CC) that falls into both categories, there are (by inclusion-exclusion) $25 + 25 - 1 = 49$ possibilities for the last pair of letters.

This gives a total of

$$2 \cdot 3 \cdot 3 \cdot 49 = \boxed{882}$$

words with the four properties.

Alternatively, you can partition this set of five-letter words into those that contain exactly one C, and those that contain exactly two C's. There are

$$2 \cdot 3 \cdot 3 \cdot 2 \cdot 24 = 864$$

of the former, and

$$2 \cdot 3 \cdot 3 = 18$$

of the latter, for a grand total of $882 = 864 + 18$.

3. Sets and Functions

Consider the function

$$s(x) = \{x\} : \{a, b, c, d\} \to \mathcal{P}(\{a, b, c, d\}).$$

(i) Is the function s injective?

Answer: Yes.
Justification: Recall that a function $f : A \to B$ is injective if it sends distinct elements of A to distinct elements of B. Formally, this means that for any $x, y \in A$, if $x \neq y$, then $f(x) \neq f(y)$. Equivalently, if $f(x) = f(y)$, then $x = y$. We shall use the latter formulation.

Suppose that $x, y \in \{a, b, c, d\}$ and $\{x\} = \{y\}$. Since $x \in \{x\}$, it follows that $x \in \{y\}$, and therefore $x = y$ because y is the only element of $\{y\}$. Said differently, if $\{x\} = \{y\}$, then these two sets have the same elements, and since the only element of $\{x\}$ is x, and the only element of $\{y\}$ is y, we conclude that $x = y$.

Note: in order to receive full credit on this problem, you had to answer correctly, and explain *why* $\{x\} = \{y\}$ implies $x = y$. Simply asserting this implication did not earn full marks. You had to talk about the elements of these sets in some way or another.

(ii) Is the function s surjective?

Answer: No.
Justification: Recall that a function $f : A \to B$ is surjective if the image of f is all of B. Formally, this means that for any $y \in B$, there is some $x \in A$ such that $f(x) = y$.

We need to show that there is an element of $\mathcal{P}(\{a, b, c, d\})$ that is not in the image of s. This follows by counting. The domain of s has 4 elements, the codomain has $2^4 = 16$ elements, and since s is a function, it assigns a unique value to every element of the domain. Therefore the function s misses at least 12 elements of the codomain. One can be more explicit. Any subset $X \subseteq \{a, b, c, d\}$ whose cardinality differs from 1 (e.g. $\varnothing, \{a, b\}, \{a, b, c\}, \{a, b, c, d\}$) is not in the image of s, because $s(x)$ is always a singleton.

Note: in order to receive full credit on this problem, you had to answer correctly, and explain *why* there are elements of $\mathcal{P}(\{a, b, c, d\})$ that are not in the image of s. Giving an example or counting were both considered acceptable justification.

4. Relations

Consider the binary relation R on $\{1, 2, 3, 4\}$ whose matrix is

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

(a) *Is R reflexive?* **No.** $1 \not R\, 1$ and $4 \not R\, 4$.
Using M: not every diagonal entry is 1.

(b) *Is R symmetric?* **No.** $2\, R\, 1$ but $1 \not R\, 2$.
Using M: M is not symmetric, i.e. we do not have $M_{ij} = M_{ji}$ for all $i, j \in \{1, 2, 3, 4\}$. In particular, $M_{12} \neq M_{21}$.

(c) *Is R antisymmetric?* **Yes.** The only pairs (i, j) for which both $i\, R\, j$ and $j\, R\, i$ hold are $(2, 2)$ and $(3, 3)$. Both of these pairs satisfy $i = j$.
In terms of the matrix, whenever $i \neq j$ and $M_{ij} = 1$, we have $M_{ji} = 0$.

Note that, while in this case it is also true that whenever $i \neq j$ and $M_{ij} = 0$ we have $M_{ji} = 1$, this is not relevant to antisymmetry.

(d) *Is R transitive?* **Yes.** Using M is the best way to go here. We have

$$M^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & 1 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 \end{pmatrix}.$$

Every nonzero entry of M^2 is also nonzero in M (i.e. $(M^2)_{ij} \neq 0$ implies $M_{ij} \neq 0$), so $R \circ R$ refines R and hence R is transitive (by homework).