DO NOT OPEN THIS EXAM UNTIL YOU ARE INSTRUCTED TO DO SO!

Class: Math 61, Lecture 1
Instructor: Jonathan Rubin
Exam: Midterm I
Date: 21 October 2019
Time: 11:00 AM – 11:50 AM

THIS IS A CLOSED BOOK EXAM. NO OUTSIDE AIDS, SUCH AS NOTES, TEXTBOOKS, CALCULATORS, OR CELLPHONES ARE PERMITTED.

First and Last Name: [redacted]

Student ID Number:

Section and Teaching Assistant: David Soukup 1A

I understand that this is a closed book exam. I certify that the following work is mine alone, and I pledge that I have neither given nor received unauthorized assistance on this test.

Signature: [redacted]

Instructions: This is a 50-minute exam. It consists of four problems, and there is an extra piece of scratch paper at the end. Please write your answers in the space provided. If you run out of room, then please continue onto the back of the page and indicate clearly that you have done so. **Good luck!**

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
Total:	40	

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1. (10 points) Prove that the equation

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \cdots + \frac{1}{n \cdot (n+1)} = \frac{n}{n+1}$$

holds for every integer $n \geq 1$.

base $n=1$: $\frac{1}{1(1+1)} = \frac{1}{1+1} \rightarrow \frac{1}{2} = \frac{1}{2}$ ✓

induction $n=n+1$: $\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \cdots + \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} = \frac{n+1}{n+2}$

Since $\frac{1}{2} + \frac{1}{6} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

$$= \frac{n}{n+1} + \frac{1}{(n+1)(n+2)}$$

$$= \frac{n(n+2)}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)}$$

$$= \frac{n^2+2n+1}{(n+1)(n+2)}$$

$$= \frac{(n+1)^2}{(n+1)(n+2)}$$

$$= \frac{n+1}{n+2}$$ ✓

2. (10 points) Consider the function

$$s(x) = \{x\} : \{a, b, c, d\} \to \mathcal{P}(\{a, b, c, d\}),$$

where $\mathcal{P}(\{a, b, c, d\})$ is the set of all subsets of $\{a, b, c, d\}$. This means that the function s sends a letter $x \in \{a, b, c, d\}$ to the *singleton set* $\{x\}$, whose only element is x.

(a) (5 points) Is the function s injective? Justify your answer.

S is injective as for any $x \in \{a,b,c,d\}$ $s(x) = \{x\}$, therefore any value $x \in \{a,b,c,d\}$ can only map to itself and therefore it is the only unique value that maps to itself (one-to-one).

(b) (5 points) Is the function s surjective? Justify your answer.

The function is not surjective as it does not map to every subset of $\{a,b,c,d\}$, it only maps to the subsets $\{a\}$, $\{b\}$, $\{c\}$, and $\{d\}$.

3. (10 points) Consider the binary relation R on $\{1, 2, 3, 4\}$ whose matrix is

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}.$$

This means that iRj is true if the (i, j)-entry is 1, and false if it is 0.

(a) (2 points) Is the relation R reflexive? Briefly explain.

The relation is not reflexive as not all $x \in \{1,2,3,4\}$ satisfy xRx in this matrix (example: $(4,4) \notin R$).

(b) (2 points) Is the relation R symmetric? Briefly explain.

The relation is not symmetric as the matrix is not symmetric along the main diagonal and, for all $x, y \in \{1,2,3,4\}$, it is not always true that xRy and yRx.
For example, $(3,1) \in R$, but $(1,3) \notin R$.

(c) (3 points) Is the relation R antisymmetric? Briefly explain.

The relation R is antisymmetric as for any $x, y \in \{1,2,3,4\}$ such that xRy and yRx, then $x=y$.

(d) (3 points) Is the relation R transitive? Briefly explain.

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 1 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} = R$$

R is transitive as for any $x, y, z \in \{1,2,3,4\}$ such that xRy and yRz, then xRz.

4. (10 points) How many 5-letter (possibly nonsense) words w are there in $\{A, B, C, \ldots, Z\}$ that satisfy the following four properties?

↳ order matters

(i) The word w starts with an A or B, and

(ii) the word w ends with an X, Y, or Z, and

(iii) the word w contains at least one C, and

(iv) the word w contains exactly one D.

Please simplify your answer as much as possible.

$\binom{2}{1}$ ___ ___ ___ ___ $\binom{3}{1}$ ___

2 option — at least one C, exactly one D — 3 options

if at least one C & exactly one D then:

1C, 1D, & 1 random

or 2C & 1D

3! as for 3 unique permutations: CDA, CAD, DCA, DAC, ACD, ADC

permutations ↓ letter for each perm ↓

1C, 1D, & 1 random: 1 option, 1 option, 26 options $= 3! \cdot 26$

2C, 1D: 2 of 1 & 1 option → $\frac{3!}{2!\,1!} = 3$ → CCD, CDC, DCC

Sum $= ((3! \cdot 26) + 3) = ((6 \cdot 26) + 3) = 159$

total words $= 2 \cdot ((3! \cdot 26) + 3) \cdot 3$

$= 2 \cdot (156 + 3) \cdot 3$

$= 2 \cdot (159) \cdot 3$

$= 318 \cdot 3 = \boxed{954}$

Extra scratch paper.