19F-MATH61-1 Midterm I B

[redacted]

TOTAL POINTS

39 / 40

QUESTION 1

1 Counting 9 / 10

+ **7 pts** Mostly correct, but double-counted those with two C's and one D, or otherwise did not consider the difference between DCE and DCC.

✓ + **10 pts Complete, correct solution**

✓ - **1 pts Secondary counting mistake**

+ **1 pts** Got (i) right - we're looking for things that satisfy all

+ **1 pts** Got (ii) right - we're looking for things that satisfy all

+ **1 pts** Got (iii) right - we're looking for things that satisfy all

+ **1 pts** Got (iv) right - we're looking for things that satisfy all

+ **0 pts** Blank / no answer / no progress towards a solution

1 by your count the last letter can't be C either

QUESTION 2

Relations 10 pts

2.1 Symmetry 2 / 2

✓ + **1 pts Correct answer**

✓ + **1 pts Good explanation**

+ **0 pts** Incorrect

+ **0 pts** Insufficient explanation

+ **0 pts** Incorrect definition

2.2 Reflexivity 2 / 2

✓ + **1 pts Correct answer**

✓ + **1 pts Good explanation**

+ **0 pts** Insufficient explanation

+ **0 pts** Incorrect

2.3 Transitivity 3 / 3

✓ + **2 pts Correct answer**

✓ + **1 pts Good explanation**

+ **0 pts** Insufficient explanation

+ **0 pts** Incorrect

+ **2 pts** Correct reasoning based on errors in matrix multiplication

2.4 Antisymmetry 3 / 3

✓ + **2 pts Correct answer**

✓ + **1 pts Good explanation**

+ **0 pts** Incorrect answer

+ **0 pts** Incorrect definition

+ **0 pts** Insufficient explanation

+ **1 pts** Correct definition

QUESTION 3

Functions 10 pts

3.1 Surjectivity 5 / 5

✓ + **3 pts Correct answer**

+ **0 pts** Incorrect answer

✓ + **2 pts Sufficient justification**

+ **1 pts** Some justification

+ **0 pts** Insufficient justification

+ **1 pts** You know the definition of surjectivity, but have not applied it correctly.

3.2 Injectivity 5 / 5

✓ + **3 pts Correct answer**

✓ + **2 pts Good justification**

+ **1 pts** Some justification

+ **0 pts** Insufficient or incorrect justification

+ **0 pts** Incorrect answer or nothing written

+ **1 pts** You know the definition of injectivity, but have not applied it correctly.

QUESTION 4

4 Induction 10 / 10

✓ + **3 pts** **Correct base case.**

✓ + **5 pts** **Correct argument in the induction step.**

✓ + **2 pts** **Written in sentences, chains of equalities, good style.**

- **1 pts** You can't start the induction proof with the equality you are trying to prove. Write one long chain of equalities instead.

- **1 pts** Slight misunderstanding of induction

DO NOT OPEN THIS TEST UNTIL YOU ARE INSTRUCTED TO DO SO!

Class:	Math 61, Lecture 1
Instructor:	Jonathan Rubin
Exam:	Midterm I
Date:	21 October 2019
Time:	11:00 AM – 11:50 AM

THIS IS A CLOSED BOOK EXAM. NO OUTSIDE AIDS, SUCH AS NOTES, TEXTBOOKS, CALCULATORS, OR CELLPHONES ARE PERMITTED.

First and Last Name: [redacted]

Student ID Number: [redacted]

Section and Teaching Assistant: [redacted]

I understand that this is a closed book exam. I certify that the following work is mine alone, and I pledge that I have neither given nor received unauthorized assistance on this test.

Signature: [redacted]

Instructions: This is a 50-minute exam. It consists of four problems, and there is an extra piece of scratch paper at the end. Please write your answers in the space provided. If you run out of room, then please continue onto the back of the page and indicate clearly that you have done so. **Good luck!**

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
Total:	40	

DO NOT OPEN THIS TEST UNTIL YOU ARE INSTRUCTED TO DO SO!

1. (10 points) How many 5-letter (possibly nonsense) words w are there in $\{A, B, C, \ldots, Z\}$ that satisfy the following four properties?

(i) The word w starts with an A or B, and

(ii) the word w ends with an X, Y, or Z, and

(iii) the word w contains at least one C, and

(iv) the word w contains exactly one D.

Please simplify your answer as much as possible.

1st letter: 2 possible

2 _ _ _ _ 3

one D

last letter: 3 possible

3 slots, 1 D: $\binom{3}{1}$ possibilities

2 remaining slots, at least 1 C: $\binom{2}{2} + \binom{2}{1}\binom{25}{1}$ possibilities

not C second letter

total $= 2 \cdot 3 \cdot \binom{3}{1} \cdot \left[\binom{2}{2} + \binom{2}{1}\binom{25}{1}\right]$

duplicates?

Clearer expl.

5 slots: _ _ _ _ _ (1 … 5)

Slot 1: A or B. 2 choices.

Slot 5: X, Y, or Z. 3 choices.

Middle 3 slots: one is D. $\binom{3}{1}$ choices.

Remaining 2 slots: at least one C.

2 C's: $\binom{2}{2}$ choices

1 C: pick which slot is C then pick one letter that is not D

$= \binom{2}{1}\binom{25}{1}$

note: all cases are nicely disjoint no possible redundant cases

apply mult. principle!

Total words with conditions $= 2 \cdot 3 \cdot \binom{3}{1} \cdot \left[\binom{2}{2} + \binom{2}{1}\binom{25}{1}\right]$

$= 2 \cdot 3 \cdot 3 \left[1 + 50\right] = 18 \cdot 51$ possible words

2. (10 points) Consider the binary relation R on $\{1, 2, 3, 4\}$ whose matrix is

name this matrix M

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

1R2 2R3 1R?

This means that iRj is true if the (i, j)-entry is 1, and false if it is 0.

(a) (2 points) Is the relation R symmetric? Briefly explain.

Symmetric relation: $xRy \overset{\text{implies}}{\Rightarrow} yRx$.

We check to see if $M_{i,j} = M_{j,i}$ for all entries.

We see 1R2 but 2 $\not R$ 1 so **R is not symmetric**

(b) (2 points) Is the relation R reflexive? Briefly explain.

Reflexive relation: for all $x \in \{1,2,3,4\}$, xRx.

→ We see the diagonal of matrix M for relation R is not all 1's.

→ 1 $\not R$ 1, so **R is not reflexive**

(c) (3 points) Is the relation R transitive? Briefly explain.

R is transitive iff $R \circ R$ refines R.

$$R \circ R = MM = \begin{pmatrix} 0&0&0&0 \\ 1&1&1&0 \\ 1&0&1&0 \\ 1&1&1&0 \end{pmatrix}\begin{pmatrix} 0&0&0&0 \\ 1&1&1&0 \\ 1&0&1&0 \\ 1&1&1&0 \end{pmatrix} = \begin{pmatrix} 0&0&0&0 \\ 2&1&2&0 \\ 1&0&1&0 \\ 2&1&2&0 \end{pmatrix} \overset{\text{nonzeros into 1}}{\Rightarrow} \begin{pmatrix} 0&0&0&0 \\ 1&1&1&0 \\ 1&0&1&0 \\ 1&1&1&0 \end{pmatrix}$$

We see $x(R \circ R)y$ implies xRy

thus **R is transitive**

(d) (3 points) Is the relation R antisymmetric? Briefly explain.

Antisymmetric: xRy and yRx implies $x = y$

In the matrix, we check $M_{i,j} \neq M_{j,i}$ for all $i \neq j$

and $M_{i,j} = M_{j,i}$ for $i = j$ (always true; same entry in M)

→ we see all entries do fulfill this condition (visually, entries across the diag from each other are opposite values)

⇒ **R is antisymmetric**

3. (10 points) Consider the function

$$s(x) = \{x\} : \underbrace{\{a, b, c, d\}}_{X \text{ (big)}} \to \underbrace{\mathcal{P}(\{a, b, c, d\})}_{Y},$$

where $\mathcal{P}(\{a, b, c, d\})$ is the set of all subsets of $\{a, b, c, d\}$. This means that the function s sends a letter $x \in \{a, b, c, d\}$ to the *singleton set* $\{x\}$, whose only element is x.

(a) (5 points) Is the function s surjective? Justify your answer.

→ Surjective functions map $x \in X \to y \in Y$ such that all $y \in Y$ are mapped to by some $x \in X$.

→ We observe function s only maps to singleton sets, i.e. sets containing one letter.

→ Since $P(\{a,b,c,d\})$ contains non-singleton sets (sets containing more than 1 letter), we find many counterexamples to the surjectivity of s.

→ one counterexample: $\{a,b\} \in P(\{a,b,c,d\})$, but s does not map to $\{a,b\}$.

→ we conclude function s is not surjective

(b) (5 points) Is the function s injective? Justify your answer.

→ Injective functions map $x \in X \to y \in Y$ such that no $y \in Y$ is mapped to by multiple $x \in X$ (one to one).

→ Suppose $s(x_1) = s(x_2)$. Then $\{x_1\} = \{x_2\}$, or $x_1 = x_2$ (they are the same element).
This implies each y in the range of function f is mapped to by a single x.

→ we conclude function s is injective

4. (10 points) Prove that the equation

$$\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\cdots+\frac{1}{n\cdot(n+1)}=\frac{n}{n+1}$$

holds for every integer $n \geq 1$.

① Base case: $n=1$

$\frac{1}{1(1+1)} = \frac{1}{1+1}$ holds, so the equation holds.

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② Inductive step: We assume $\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}$.

Then for the $(n+1)^{th}$ case, we hope to show

$\underbrace{\frac{1}{2}+\frac{1}{6}+\ldots+\frac{1}{n(n+1)}+\frac{1}{(n+1)(n+2)}}_{LHS} = \underbrace{\frac{n+1}{(n+2)}}_{RHS}$,

LHS: we substitute our assumption to obtain

$\frac{n}{n+1}+\frac{1}{(n+1)(n+2)}$

$=\frac{n(n+2)}{(n+1)(n+2)}+\frac{1}{(n+1)(n+2)} = \frac{n^2+2n+1}{(n+1)(n+2)} = \frac{(n+1)^2}{(n+1)(n+2)}$

$=\frac{n+1}{n+2}$

$=$ RHS

So we have shown that if $\frac{1}{2}+\frac{1}{6}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}$,

then $\frac{1}{2}+\frac{1}{6}+\ldots+\frac{1}{n(n+1)}+\frac{1}{(n+1)(n+1+1)}=\frac{n+1}{(n+1+1)}$

Thus from ① and ②, we have shown

$\frac{1}{2}+\frac{1}{6}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}$ holds for every integer $n \geq 1$.

Extra scratch paper.