Midterm 1 (Version A)

UCLA: Math 61, Winter 2018

Instructor: Jens Eberhardt
Date: *02 February 2017*

- This exam has 4 questions, for a total of 36 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- Indicate your final answer clearly.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.

Name: ______________________________

ID number: ______________________________

Discussion section (please circle):

Day/TA	HUNT, CHRISTOPHER	HAN, KYUTAE	MENEZES, DEAN
Tuesday	1A	1C	1E
Thursday	1B	1D	1F

Question	Points	Score
1	12	12
2	10	8
3	6	6
4	8	7
Total:	36	33

20

13

Please note! The following two pages will not be graded. You must indicate your answers here for them to be graded!

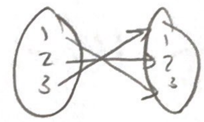
Question 1.

Part	A	B	C	D
(a)	X			
(b)	X			
(c)			X	
(d)			X	
(e)				X
(f)			X	

1. Each of the following questions has exactly one correct answer. Choose from the four options presented in each case. No partial points will be given.

(a) (2 points) Let $X = \{1,2,3\}$, then $R = \{(1,3),(2,2),(3,1)\}$ is

A. bijective
B. reflexive
C. antisymmetric
D. not symmetric



(b) (2 points) Let $X = \mathbb{Z}_{\geq 0} = \{0,1,2,\dots\}$. Define an equivalence relation R on X by:

$$xRy \text{ if } x - y \text{ is divisible by } 3.$$

Then the partition $\mathcal{P}_R$ associated to the relation R is:

A. $\mathcal{P}_R = \{\{0,3,6,\dots\},\{1,4,7,\dots\},\{2,5,8,\dots\}\}$
B. $\mathcal{P}_R = \{\{0,2,4,\dots\},\{1,3,5,\dots\}\}$
C. $\mathcal{P}_R = \{\{0\},\{1\},\{2\},\dots\}$
D. $\mathcal{P}_R = \{(0,0),(0,3),(3,0)\dots,(3,3),(3,6),(6,3),\dots,$
$(1,1),(1,4),(4,1),\dots,(4,4),(4,7)(7,4),\dots,$
$(2,2),(2,5),(5,2),\dots,(5,5),(5,8)(8,5)\dots\}$

(c) (2 points) Define a partition $\mathcal{P}$ on $\{0,1,2,3\}$ by

$$\mathcal{P} = \{\{0\},\{1,3\},\{2\}\}.$$

Let $R_{\mathcal{P}}$ be the associated equivalence relation on $\{0,1,2,3\}$. Then

A. $R_{\mathcal{P}} = \{(0,0),(1,3),(3,1),(2,2)\}$
B. $R_{\mathcal{P}} = \{\{0\},\{1,3\},\{2\}\}$
C. $R_{\mathcal{P}} = \{(0,0),(1,1),(2,2),(3,3),(1,3),(3,1)\}$
D. $R_{\mathcal{P}} = \{0,1,2,3\}$

(0,0), (1,1), (1,3), (3,1), (2,2).

(d) (2 points) Let $X = \{0, 1, 2, 3\}$. For a set Y denote by $\mathcal{P}(Y) = \{S | S \text{ is a subset of } Y\}$ the power set of X. Then

A. $|\mathcal{P}(X \times X)| = 2 \cdot 16$
B. $|\mathcal{P}(X \times X)| = 2^8$
C. $|\mathcal{P}(X \times X)| = 2^{16}$
D. $|\mathcal{P}(X \times X)| = 2 \cdot 8$

$|X \times X| = 16$

$|P(X \times X)| = 2^{16}$

(e) (2 points) Let $n \geq 1$ be a positive integer. Then

$$\prod_{i=1}^{n} 2i$$

is equal to

A. $2^n \frac{n(n+1)}{2}$
B. $\frac{2^n 2^{n+1}}{2}$
C. $(2n)!$
D. $2^n \cdot n!$

$= 2 \times 4 \times 6 \times \cdots \times 2n$

$= 2^n(1 \times 2 \times 3 \times \cdots \times n) = 2^n n!$

$= 2^n \prod_{i=1}^{n} i$

$= \cancel{(2n)}!$

$2n \cdot (2n-1) \cdot (2n-2) \cdots 1$

(f) (2 points) Let $X = \{a, b\}$. Denote by $X^{\geq 3}$ the set of all strings over X of length bigger or equal than three and by X^* the set of all strings over X. Then

A. $|X^* - X^{\geq 3}| = 5$
B. $|X^* - X^{\geq 3}| = 6$
C. $|X^* - X^{\geq 3}| = 7$
D. $|X^* - X^{\geq 3}| = 8$

$X^* = \{\varepsilon, a, aa, aaa, \cdots\}$

$X^{\geq 3} = \{aaa, abc, \cdots\}$

$= X^{<3} = \{\varepsilon, a, b, aa, bb, ab, ba\}$

2. In the following questions, simply write down your answer. There is *no justification needed.* Do not simplify expressions as $2^4, 6!, C(n,r), \ldots$.

(a) (2 points) Determine the number of 7-bit strings starting in $\underbrace{1010}_{4}$.

$\boxed{2^3}$

?

(b) (2 points) Determine the number of 7-bit strings ending in 1010.

$\boxed{2^3}$

(c) (2 points) Determine the number of 7-bit strings starting in 1010 or ending in 1010.

$2^3 + 2^3 - 0 = \boxed{2^3 + 2^3}$

(d) (2 points) You have three friends: Rocco, Gina and Hans. And you have seven different sweets: a popsicle, a piece of apple pie, a chocolate bar, a Berliner, a jelly doughnut, a marshmallow and a lemon drop.
You want to give two sweets to Rocco, three to Gina and two to Hans. In how many ways could you do this?

Step 1: Give two sweets to Rocco $= C(2+(7-1), 7-1) = C(8,6)$
Step 2: " " three " " Gina $= C(3+(5-1), 5-1) = C(7,4)$
Step 3: " " two " " Hans $= C(2+(2-1), 2-1) = C(3,1)$

$= \boxed{C(8,6) \cdot C(7,4) \cdot C(3,1)}$

(e) (2 points) Let $X = \{1, 2, 3, 4, 5\}$. Determine the number of elements of the following set

$$\{S \mid S \text{ is a subset of } X \text{ and } |S| = 3\}.$$

$\boxed{C(5,3)}$

3. (6 points) Prove by induction that

$$\sum_{i=1}^{n}(2i-n)=n$$

for any integer $n \geq 1$.
Hint: Use

$$\sum_{i=1}^{n+1}(2i-(n+1)) = \left(\sum_{i=1}^{n+1}(2i-n)\right)-(n+1).$$

Claim: $S_n = (2-n)+(4-n)+(6-n)+\cdots+(2n-n)=n$

Basis Step: $(n=1)$

$2-n=n \rightarrow 2-1=1 \rightarrow 1=1$ ✓

S_n is true for $n=1$.

Inductive Step:

Assume $S_n = n = \sum_{i=1}^{n}(2i-n)$

Then $S_{n+1} = \sum_{i=1}^{n+1}(2i-(n+1)) = \left(\sum_{i=1}^{n+1}(2i-n)\right)-(n+1)$

$= S_n + 2(n+1) - 2(n+1) - (n+1)$

$= S_n + 1$

$= n+1$

Hence, by mathematical induction, the statement is true.

4. Answer the following questions, *justifying* your answers. (If an answer is Yes, explain why. If an answer is No, give a counterexample.)

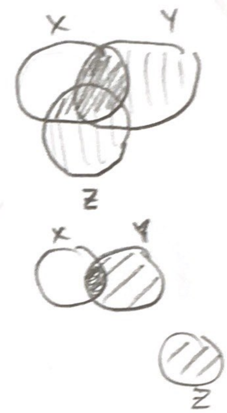
(a) (2 points) Define $f : \mathbb{R} \to \mathbb{R}_{\geq 0}, x \mapsto x^2$. Is f surjective?

Yes, since for all $y \in \mathbb{R}_{\geq 0}$, there is an $x \in \mathbb{R}$ such that $f(x) = y$.
Namely, $f(x) = x^2$, which covers all $\mathbb{R}_{\geq 0}$.

-1

(b) (2 points) Let X, Y, Z be sets. Then

$$X \cap (Y \cup Z) = (X \cap Y) \cup Z.$$



No. Counterexample:

$X = \{1, 2, 3\}$ $\quad X \cap (Y \cup Z) = \{2, 3\}$

$Y = \{2, 3, 4\}$ $\quad (X \cap Y) \cup Z = \{2, 3, 5, 6\}$

$Z = \{5, 6\}$ $\quad \{2, 3\} \neq \{2, 3, 5, 6\}$

(c) (2 points) Let $X = \{1, 2, 3\}$ and $Y = \{a, b, c\}$. Is there a bijective function $f : X \to Y$?

Yes, for example: $\{(1, a), (2, b), (3, c)\}$

This is injective because for all $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.
This is surjective because for all $y \in Y$, there is an $x \in X$ such that $f(x) = y$.

(d) (2 points) Let $X = \{a, b, c, \ldots, z\}$ be the alphabet. Let α and β be strings over X. Is

$$\alpha\beta = \beta\alpha?$$

No. Counterexample:

$\alpha = ab$ $\quad \alpha\beta = abyz$

$\beta = yz$ $\quad \beta\alpha = yzab$

$abyz \neq yzab$

This page has been left intentionally blank. You may use it as scratch paper. It will not be graded unless indicated very clearly here and next to the relevant question.