MIDTERM 2 (MATH 61)

FRIDAY, MAY 17TH

Name: Justin Woo

ID: __________

Circle your discussion section:

Tuesday	Thursday	
2A	2B	TA: Harris Khan
2C	(2D)	TA: Fred Vu
2E	2F	TA: Matthew Stone

This exam has 8 pages, including the cover page. Please make sure your exam includes each page. Please write your name on *each* page you submit. You will have 50 minutes to complete this exam. **You may not use a calculator**, or consult your textbook, class notes, or any other materials. If you need scratch paper or more space for your answers, please use the back of the pages.

If there is any work on the backs of the pages which you would like to have graded, please indicate this clearly on the front of the page for the corresponding problem.

Show your work for these problems, don't just give an answer. If a question asks you to prove something, please write a complete proof. Unless otherwise stated, you may use any results proved in class or in the textbook, but please make it clear when you are doing so. Unless otherwise stated, you will *not* receive full credit for giving the correct answer with no explanation. You may still earn partial credit even if your final answer is incorrect.

Remember that you are bound by a conduct code, and that you may not look at anyone's paper or let anyone look at your paper.

Question	Points	Score
1	10	
2	30	
3	20	
4	25	
5	15	
Total:	100	

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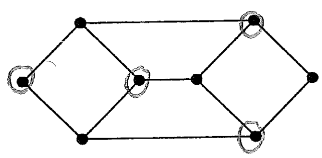
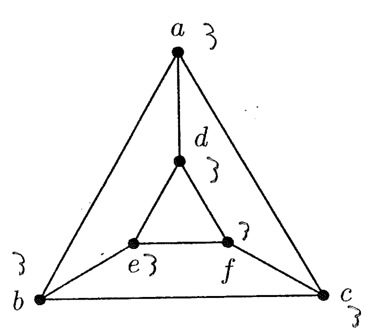
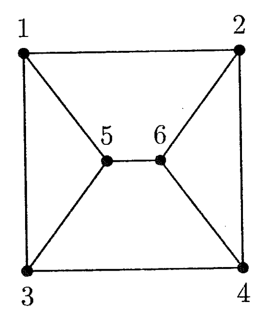
1. [10 pts, 2 points each] Mark each of the following statements as either TRUE or FALSE. For this question you do not need to show any work beyond the final answer.

Be sure to read the questions carefully!

(a) The complete bipartite graph $K_{61,2019}$ has an Euler cycle.	F
(b) There is a graph with 9 vertices in which every vertex has degree 3.	F
(c) The graph G with adjacency matrix $A = \begin{pmatrix} 2 & 0 & 1 & 0 & 3 \\ 0 & 4 & 0 & 3 & 0 \\ 1 & 0 & 6 & 0 & 5 \\ 0 & 3 & 0 & 2 & 0 \\ 3 & 0 & 5 & 0 & 8 \end{pmatrix}$ is connected.	T
(d) The graph below is bipartite: 	T
(e) The two graphs below are isomorphic:  	T

Name:________________________

2. [30 pts] Solve the following recursion relations. In each case, your answer should be a formula for the n^{th} term of the sequence in terms of n.

Show your work. You may use results proved in class or in the textbook, but make it clear how you are getting your answers. A correct answer on its own will not be sufficient for full credit.

(a) [6 pts] Find a *general* solution to the recursion relation $a_n = 4a_{n-1} + 5a_{n-2}$. (That is, find a formula for a_n in terms of n and some unknown constants, that will work for any choice of initial conditions.)

The characteristic polynomial is $t^2 - 4t - 5 = 0$. We find the roots.

$(t-5)(t+1) = 0$

$t = -1, 5$

A general solution is

$a_n = b(-1)^n + d(5)^n$ for some $b, d \in \mathbb{R}$.

(b) [4 pts] Find the solution to the recurrence relation $a_n = 4a_{n-1} + 5a_{n-2}$ with initial conditions $a_0 = 5$ and $a_1 = 7$.

We use the initial conditions to find a particular solution.

$a_0 = b(-1)^0 + d(5)^0$

$a_0 = b + d = 5$

$b = 5 - d$

$a_1 = b(-1)^1 + d(5)^1$

$a_1 = -b + 5d = 7$

$b = 5d - 7$

$5 - d = 5d - 7$

$5 = 6d - 7$

$12 = 6d$

$d = 2$

$b = 5 - 2 = 3$

So the particular solution is

$a_n = 3(-1)^n + 2(5)^n$.

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(c) [10 pts] Find the solution to the recurrence relation $b_n = b_{n-1}^4 b_{n-2}^5$ with initial conditions $b_0 = 1$ and $b_1 = 16$.

Let $a_n = \log_2 b_n$.

$\log_2 b_n = \log_2(b_{n-1}^4 b_{n-2}^5)$

$a_n = \log_2 b_{n-1}^4 + \log_2 b_{n-2}^5$

$a_n = 4\log_2 b_{n-1} + 5\log_2 b_{n-2}$

$a_n = 4a_{n-1} + 5a_{n-2}$

The characteristic polynomial for a_n is

$t^2 - 4t - 5 = 0$.

$(t-5)(t+1) = 0$

$t = -1, 5$

So $a_n = b(-1)^n + d(5)^n$.

Using the initial conditions

$a_0 = \log_2 b_0 = \log_2 1 = 0$

$a_1 = \log_2 b_1 = \log_2 16 = 4$

$a_0 = b(-1)^0 + d(5)^0$

$a_0 = b + d = 0$

$b = -d$

$a_1 = b(-1)^1 + d(5)^1$

$a_1 = -b + 5d = 4$

$b = 5d - 4$

$-d = 5d - 4$

$4 = 6d$

$d = \frac{2}{3}$ $b = -\frac{2}{3}$

We have

$a_n = \log_2 b_n = -\frac{2}{3}(-1)^n + \frac{2}{3}(5)^n$

so

$b_n = 2^{a_n}$

$b_n = 2^{(-\frac{2}{3}(-1)^n + \frac{2}{3}(5)^n)}$

(d) [10 pts] Find the solution to the recurrence relation $c_n = 4c_{n-1} + 5c_{n-2} + 16$ with initial conditions $c_0 = 0$ and $c_1 = 2$.

We first ignore the forcing term 16 and solve the homogeneous relation

$c_n = 4c_{n-1} + 5c_{n-2}$

The characteristic polynomial is

$t^2 - 4t - 5 = 0$

$(t-5)(t+1) = 0$

$t = -1, 5$

So the homogeneous solution is

$c_n = b(-1)^n + d(5)^n$.

The forcing term is a constant so we assume $c_n = k$ for some constant k.

$c_n = k$

$c_{n-1} = k$

$c_{n-2} = k$

Substituting into the original relation,

$k = 4k + 5k + 16$

$k = 9k + 16$

$8k = -16$

$k = -2$

The general solution is

$c_n = b(-1)^n + d(5)^n - 2$.

We use the initial conditions to find a particular solution.

$c_0 = b(-1)^0 + d(5)^0 - 2$

$c_0 = b + d - 2 = 0$

$b + d = 2$

$b = 2 - d$

$c_1 = b(-1)^1 + d(5)^1 - 2$

$c_1 = -b + 5d - 2 = 2$

$-b + 5d = 4$

$b = 5d - 4$

$2 - d = 5d - 4$

$2 = 6d - 4$

$6 = 6d$

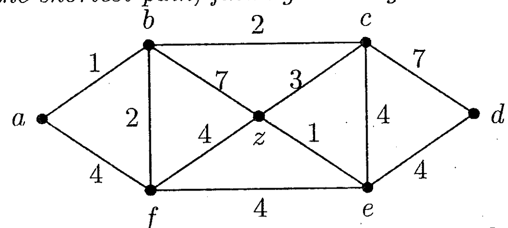
$d = 1$

$b = 2 - 1 = 1$

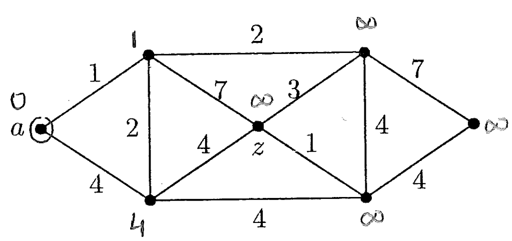
The particular solution is

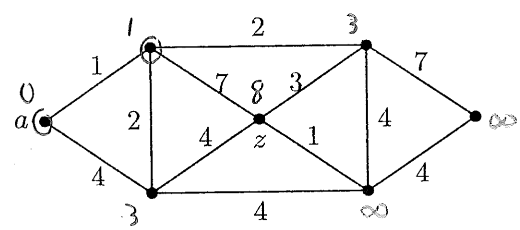
$c_n = (-1)^n + (5)^n - 2$.

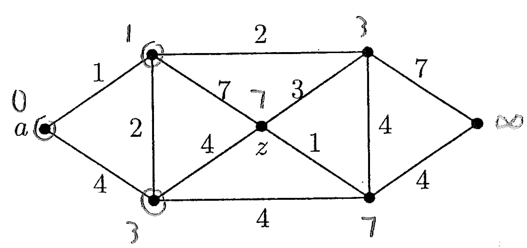
3. [20 pts] Use Dijkstra's algorithm to find the length of the shortest path (i.e. the path for which the sum of the labels is as small as possible) between a and z in the weighted graph below. *You do not need to find the shortest path, finding it's length will be sufficient.*

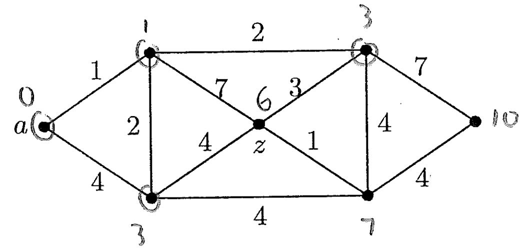


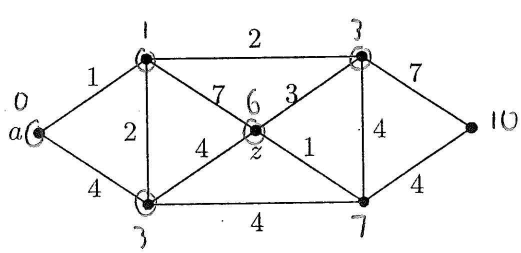
Show each step of Dijkstra's algorithm. A correct final answer with no work shown will not be sufficient for full credit. Use the blank graphs below for your answer. If you make a mistake, clearly cross it out and continue using the next blank graph. There are additional blank graphs on the back of this page.

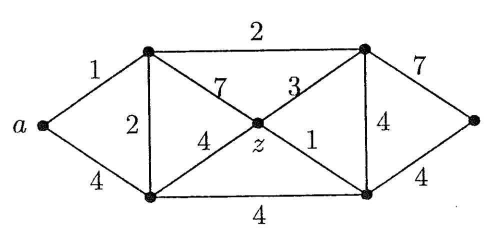






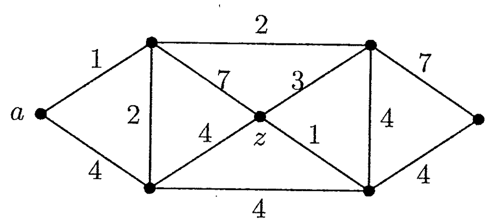






Answer: 6

Check this box if you used any graphs from the back of the page: ☐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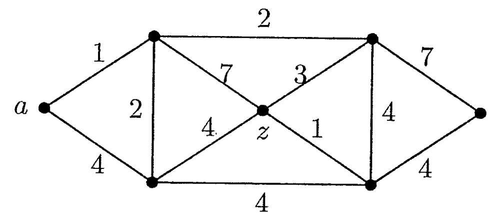
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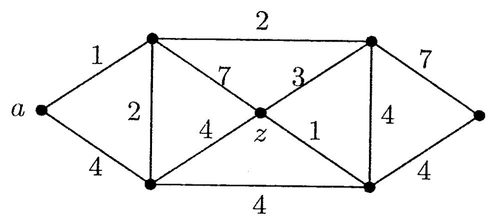
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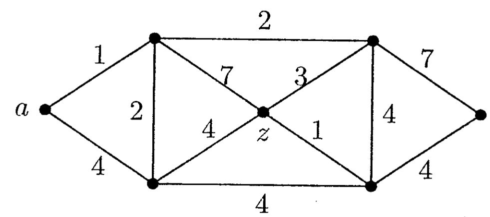
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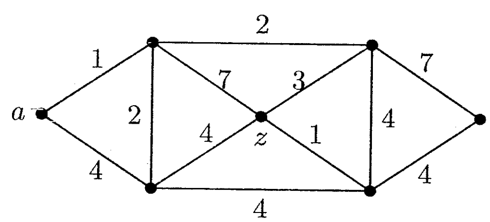
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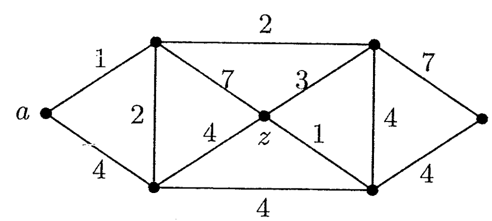
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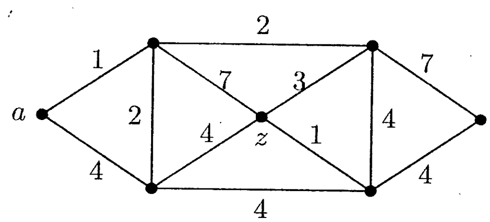
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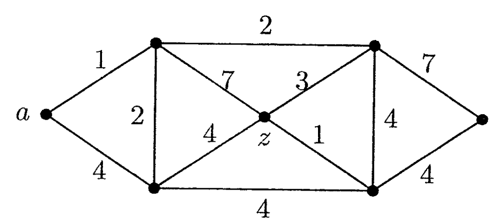
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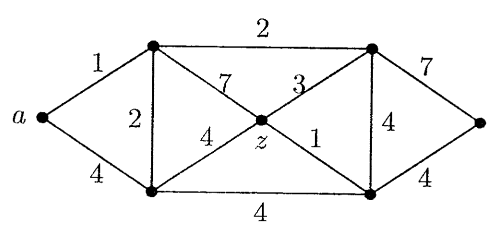
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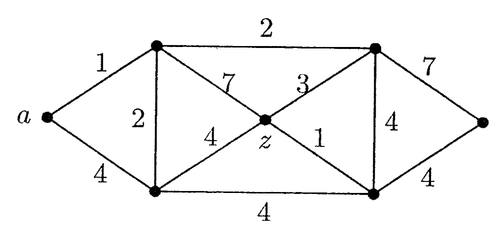
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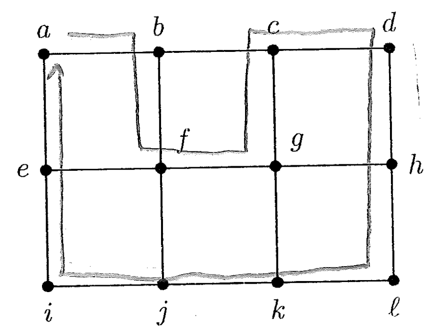
Name:______________________________

4. [25 pts] In each of the following graphs, either find a Hamiltonian cycle (i.e. a cycle which uses every vertex exactly once), or prove that the graph does not have a Hamiltonian cycle.

If the graph does have a Hamiltonian cycle, **CLEARLY** drawing this cycle on the provided graph (so that there's no ambiguity as to which edges are used, and in which order), or listing out the vertices in the order traveled (i.e. writing something like (a, b, e, d, c, a)) will be sufficient for full credit.

If the graph does not have a Hamiltonian cycle, you must give an explanation as to why. *Simply drawing diagrams with no explanation will NOT be sufficient for full credit.*

(a) [10 pts]



A Hamiltonian cycle for this graph is
(a, b, f, g, c, d, h, ℓ, k, j, i, e, a).

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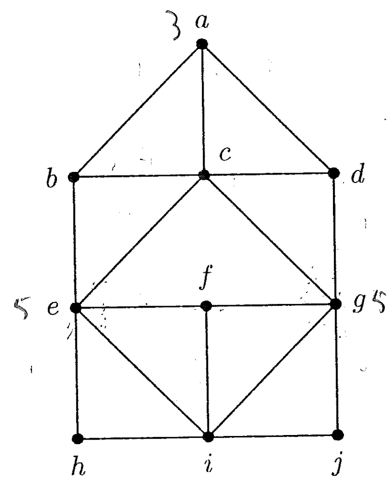
(b) [15 pts]



This graph has 18 edges and 10 vertices.
A Hamiltonian cycle has an equal number of edges and vertices
so this graph needs 10 edges for a Hamiltonian cycle.

Every vertex in a Hamiltonian cycle has degree 2.
Vertices e and g have degree 5, while vertex a has degree 3.
We must remove 7 edges from the graph, leaving us with 11 edges.

Name:______________________________

5. [15 pts]

(a) [10 pts] Let G be a *simple* graph with 10 vertices, in which every vertex has degree at least 5. Prove that G is connected. [Hint: You may want to use the pigeonhole principle.]

If every vertex has degree 5, then there are 50/2 = 25 edges in G. There are 25 edges in G and 10 vertices.

Suppose G is not connected. Then, there exist at least two vertices with no edges between them.
By the Pigeonhole Principle this cannot hold as there are more edges than vertices.

Therefore G is connected.

(b) [5 pts] Draw a *simple* graph with 10 vertices which is *not* connected, in which every vertex has degree 4.