Math 61 Midterm 1

David Hung Tung

TOTAL POINTS

98 / 100

QUESTION 1

1 10 / 10

✓ **+ 2 pts** a: True

✓ **+ 2 pts** b: False

✓ **+ 2 pts** c: True

✓ **+ 2 pts** d: True

✓ **+ 2 pts** e: False

QUESTION 2

35 pts

2.1 a 10 / 10

✓ **- 0 pts** Correct 3!6!

2.2 b 10 / 10

✓ **- 0 pts** Correct 3^6*2^4*10!/(4!6!)

2.3 c 15 / 15

✓ **- 0 pts** Correct

QUESTION 3

3 14 / 15

✓ **+ 5 pts** Showed reflexivity

✓ **+ 5 pts** Showed symmetry

✓ **+ 5 pts** Showed transitivity

+ 0 pts No credit due

- 1 Point adjustment

You don't explain why y/x can be expressed as a fraction.

QUESTION 4

4 20 / 20

✓ **- 0 pts** Correct

QUESTION 5

20 pts

5.1 a 9 / 10

+ 0 pts No serious progress / didn't understand the relevant definitions

+ 3 pts Tried something somewhat sensible, but did not make any serious progress towards a formal proof.

+ 6 pts Started with f(x) = f(y), applied g to both sides, concluded h(x) = h(y), but could not properly conclude the result (or struggled to articulate it formally).

✓ **+ 10 pts** **Correct. Started with f(x) = f(y), applied g to both sides, concluded h(x) = h(y), and used injectivity of h to conclude x = y. (Or, ran an equivalent proof by contradiction).**

- 1 Point adjustment

I think you basically have it, but you should clarify the last step by saying h(x) = h(y) implies x = y.

5.2 b 10 / 10

+ 0 pts No serious progress

+ 2 pts Bin 4: finite X,Y,Z, one of them is not a function, or g is injective or h is not injective or the composition is not done correctly. Or definitions missing

✓ **+ 10 pts** **Bin 5: finite X,Y,Z with f,g,h written out as sets or picture, g clearly not injective and h clearly injective**

+ 2 pts Bin 6: some complicated f,g not satisfying the basic conditions necessary for the counterexample (or not at all justified to have these properties)

+ 5 pts Bin 1: g is simple, f is complicated. things like x^2 for g, complicated function for f, X,Y,Z not specified, and h not proven to be injective and nontrivial to see that it is

+ **8 pts** Bin 2: both f,g are simple. x^2, x^1/2, and worked out why h is injective (or it is clear) and why g is not. Left out what X, Y, Z should be (or incorrectly specified them)

+ **10 pts** Bin 3: Fully correct example (X,Y,Z specified or drawn, g not injective and h injective)

Midterm 1 (Math 61)

Monday, April 22nd

Name: David Tung

ID: 505113462

Circle your discussion section:

Tuesday	Thursday	
2A	2B	TA: Harris Khan
2C	2D	TA: Fred Vu
2E	2F	TA: Matthew Stone

This exam has 7 pages, including the cover page. Please make sure your exam includes each page. Please write your name on *each* page you submit. You will have 50 minutes to complete this exam. **You may not use a calculator**, or consult your textbook, class notes, or any other materials. If you need scratch paper or more space for your answers, please use the back of the pages.

If there is any work on the backs of the pages which you would like to have graded, please indicate this clearly on the front of the page for the corresponding problem.

Show your work for these problems, don't just give an answer. If a question asks you to prove something, please write a complete proof. Unless otherwise stated, you may use any results proved in class or in the textbook, but please make it clear when you are doing so. Unless otherwise stated, you will *not* receive full credit for giving the correct answer with no explanation. You may still earn partial credit even if your final answer is incorrect.

Remember that you are bound by a conduct code, and that you may not look at anyone's paper or let anyone look at your paper.

Question	Points	Score
1	10	
2	35	
3	15	
4	20	
5	20	
Total:	100	

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1. [10 pts, 2 points each] Mark each of the following statements as either TRUE or FALSE. For this question you do not need to show any work beyond the final answer.

Be sure to read the questions carefully!

(a) The set $\{\{1\}, 3, \{2,3\}, 2, 3\}$ has cardinality 4.	TRUE
(b) Every relation is either symmetric or antisymmetric.	FALSE
(c) If R is a relation satisfying $R = R^{-1}$, then R is symmetric.	TRUE
(d) The set $f = \{(C, \clubsuit), (B, \heartsuit), (E, \heartsuit), (D, \spadesuit), (A, \clubsuit)\}$ is a function from $X = \{A, B, C, D, E\}$ to $Y = \{\diamondsuit, \clubsuit, \heartsuit, \spadesuit\}$.	TRUE
(e) The set $X = \{1, 2, 3, 4, 5, 6\}$ has more subsets of cardinality 4 than subsets of cardinality 3.	FALSE

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*I will use $\binom{n}{r}$ to denote $C(n,r)$.

2. [35 pts] Compute the following quantities. You may leave your answers in terms of exponents and factorials, but do not leave your final answers in terms of $P(n,r)$ or $C(n,r)$ (so $4^{12}\frac{15!}{3!6!2!}$ would be an acceptable final answer, but $P(10,3)C(18,7)$ would not).

Show your work. It should be clear how you got your answers.

(a) [10 pts] The number of permutations of the letters *COMPUTER* that contain the letters *CPU* together in any order (so for instance, *MTUPCREO* would be one such arrangement, but *PMTCOREU* would not).

Number of ways CPU can be rearranged is 3!.

Number of ways remaining 5 letters and the ordering of "CPU" can be rearranged is 6!.

Therefore, there are $\boxed{6!\,3!}$ permutations.

(b) [10 pts] The coefficient of x^6y^8 in the expansion of $(3x+2y^2)^{10}$. [Don't forget that y is squared in this expression.]

Let $a = 3x$ and $b = 2y^2$.

For a coefficient of x^6y^8, we are looking for the a^6b^4 term.

This is equal to $\binom{10}{4}(3x)^6(2y^2)^4 = \frac{10!}{6!4!}3^6 2^4 x^6 y^8$

$\boxed{\frac{10!}{6!4!}3^6 2^4}$ is the coefficient of x^6y^8.

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(c) [15 pts] A standard 6-sided die (with sides numbered 1-6) is rolled 10 times in a row. How many possible outcomes are there in which *exactly* four 5's were rolled, and no two 5's were ever rolled in a row.

[So for example 4562251565 would be one such outcome, but 2544551533 would not. Also order matters, so 4562251565 would be considered a different outcome from 5254152566.]

First "choose" the 6 non-5 rolls, as such:

(1) _ (2) _ (3) _ (4) _ (5) _ (6) _ (7)

This can be done in 5^6 ways (choose 1,2,3,4,6).

Then, place the 4 5's in the seven slots opened up, which can be done in $\binom{7}{4}$ ways.

Therefore, there are $5^6\binom{7}{4} = \boxed{5^6 \frac{7!}{4!3!}}$ outcomes.

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3. [15 pts] Let $\mathbb{R}^+ = \{x \in \mathbb{R} | x > 0\}$ be the set of positive real numbers. Define a relation R on $\mathbb{R}^+$ by $(x, y) \in R$ if x/y is a rational number. Prove that R is an equivalence relation.

To be an equivalence relation, R must be reflexive, symmetric, and transitive.

Choose any $x \in R^+$. Any number divided by itself is 1 (a rational number), so we know there is a relation (x, x), meaning R is reflexive.

Let the relation (x, y) exist. By definition of R, $\frac{x}{y}$ is a rational number. We see that $\frac{y}{x}$ is also a rational number since it can be expressed in the form of a fraction; therefore, for all relations (x, y) that exist, (y, x) also exists. Since there is a relation (y, x), R is symmetric.

Let the relation (x, y) and (y, z) exist. By definition of R, $\frac{x}{y}$ is a rational number and $\frac{y}{z}$ is a rational number. Multiplying two rational numbers yields another rational number, so we know $\frac{x}{y}\left(\frac{y}{z}\right) = \frac{x}{z}$ is rational. Therefore, the relation (x, z) exists and R is transitive.

Because R is reflexive, symmetric, and transitive, R is by definition, an equivalence relation.

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4. [20 pts] Prove that for any positive integer n:

$$1(1!) + 2(2!) + 3(3!) + \cdots + n(n!) = (n+1)! - 1$$

[Hint: Use induction.]

Base case: $n=1$

$n(n!) \stackrel{?}{=} (n+1)! - 1$

$1(1!) \stackrel{?}{=} (2)! - 1$

$1 = 1$ ✓

we have verified the base case.

Inductive Step:

Assume the "hypothesis" is true.

$1(1!) + 2(2!) + \ldots + n(n!) + (n+1)(n+1)! = (n+1)! - 1 + (n+1)(n+1)!$

Factor out a $(n+1)!$ term. $= (n+1)! \left[(n+1)+1\right] - 1$

$= (n+1)!(n+2) - 1$

$= (n+2)! - 1$

This is the answer we would expect if we were to let $n = n+1$ in the given equation.

Since the statement is true for the base case $n=1$ and for all $n+1$, we have proved the statement, by induction, for all positive integer n.

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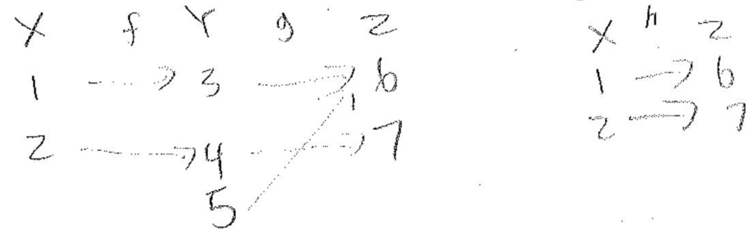
5. [20 pts] Let X, Y and Z be sets, and let $f : X \to Y$ and $g : Y \to Z$ be functions, and let $h = g \circ f$ be the composition of f and g. (That is, h is the function from X to Z defined by $h(x) = g(f(x))$.)

(a) [10 pts] Prove that if h is one-to-one, then f is one-to-one as well. [Hint: Assume that $f(x) = f(y)$, and try to use that to prove that $x = y$.]

If h is one-to-one, then we know each element of X is mapped to one and only one element of Z. Assume the opposite, that f is not one-to-one. This means there exist some f(x) = f(y) for which x ≠ y. Since g is a function, this means g(f(x)) = g(f(y)) in at least one instance (because f is not one-to-one). Since h is defined as h(x) = g(f(x)), this would imply h is not one-to-one, which is a contradiction. Therefore, for h to be one-to-one, f must be one-to-one as well.

(b) [10 pts] Give a counterexample to show that it is possible for h to be one-to-one while g is not one-to-one. That is, give examples of sets X, Y and Z and functions $f : X \to Y$ and $g : Y \to Z$ such that $h = g \circ f$ is one-to-one, but g is *not* one-to-one.



Let set X = {1, 2}, Y = {3, 4, 5} and Z = {6, 7}. We define the function f: X→Y as f(1) = 3 and f(2) = 4. We define the function g: Y→Z as g(3) = 6, g(4) = 7, and g(5) = 6. Therefore, g is not one-to-one, since g(3) = g(5) and 3 ≠ 5. However, h is one-to-one, because every value of X maps to a unique value in Z (h(1) = 6 and h(2) = 7).