MIDTERM 1 (MATH 61)

MONDAY, APRIL 22ND

Name: Justin Woo

ID:

Circle your discussion section:

Tuesday	Thursday	
2A	2B	TA: Harris Khan
2C	(2D)	TA: Fred Vu
2E	2F	TA: Matthew Stone

This exam has 7 pages, including the cover page. Please make sure your exam includes each page. Please write your name on *each* page you submit. You will have 50 minutes to complete this exam. **You may not use a calculator**, or consult your textbook, class notes, or any other materials. If you need scratch paper or more space for your answers, please use the back of the pages.

If there is any work on the backs of the pages which you would like to have graded, please indicate this clearly on the front of the page for the corresponding problem.

Show your work for these problems, don't just give an answer. If a question asks you to prove something, please write a complete proof. Unless otherwise stated, you may use any results proved in class or in the textbook, but please make it clear when you are doing so. Unless otherwise stated, you will *not* receive full credit for giving the correct answer with no explanation. You may still earn partial credit even if your final answer is incorrect.

Remember that you are bound by a conduct code, and that you may not look at anyone's paper or let anyone look at your paper.

Question	Points	Score
1	10	
2	35	
3	15	
4	20	
5	20	
Total:	100	

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1. [10 pts, 2 points each] Mark each of the following statements as either TRUE or FALSE. For this question you do not need to show any work beyond the final answer.

Be sure to read the questions carefully!

(a) The set $\{\{1\}, 3, \{2,3\}, 2, 3\}$ has cardinality 4.	False
(b) Every relation is either symmetric or antisymmetric.	False
(c) If R is a relation satisfying $R = R^{-1}$, then R is symmetric.	True
(d) The set $f = \{(C, \clubsuit), (B, \heartsuit), (E, \heartsuit), (D, \spadesuit), (A, \clubsuit)\}$ is a function from $X = \{A, B, C, D, E\}$ to $Y = \{\diamondsuit, \clubsuit, \heartsuit, \spadesuit\}$.	True
(e) The set $X = \{1, 2, 3, 4, 5, 6\}$ has more subsets of cardinality 4 than subsets of cardinality 3.	True

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2. [35 pts] Compute the following quantities. You may leave your answers in terms of exponents and factorials, but do not leave your final answers in terms of $P(n,r)$ or $C(n,r)$ (so $4^{12}\frac{15!}{3!6!2!}$ would be an acceptable final answer, but $P(10,3)C(18,7)$ would not).

Show your work. It should be clear how you got your answers.

(a) [10 pts] The number of permutations of the letters *COMPUTER* that contain the letters *CPU* together in any order (so for instance, *MTUPCREO* would be one such arrangement, but *PMTCOREU* would not).

We first permute CPU, giving 3! permutations.

We then consider CPU as one token so that there are 5 other tokens to permute. In total, there are 6 tokens.

There are 6! ways to permute these 6 tokens.

We conclude that there are 3! · 6! permutations that contain the letters CPU in any order.

(b) [10 pts] The coefficient of x^6y^8 in the expansion of $(3x+2y^2)^{10}$. [Don't forget that y is squared in this expression.]

The Binomial Theorem states $(a+b)^n = \sum_{i=0}^{n} C(n,i)\, a^{n-i} b^i$

Let $a = 3x$ and $b = 2y^2$.

We want to find the term $i = 4$ for $n = 10$.

We compute

$C(10,4)(3x)^{10-4}(2y^2)^4$

$= \frac{10!}{4!6!}(3x)^6(2y^2)^4$

$= 3^6 2^4 \frac{10!}{4!6!} x^6 y^8$

The coefficient of x^6y^8 is $3^6 2^4 \frac{10!}{4!6!}$.

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(c) [15 pts] A standard 6-sided die (with sides numbered 1-6) is rolled 10 times in a row. How many possible outcomes are there in which *exactly* four 5's were rolled, and no two 5's were ever rolled in a row.

[So for example 4562251565 would be one such outcome, but 2544551533 would not. Also order matters, so 4562251565 would be considered a different outcome from 5254152566.]

If we want exactly four 5's to be rolled, the other six rolls must not be 5's. There are 5^6 possibilities in this case.

We also want no two 5's to be rolled in a row.

We first place the 5's.

$\underset{\times 3}{\_}\ 5\ N\ \underset{\times 3}{\_}\ 5\ N\ \underset{\times 3}{\_}\ 5\ N\ \underset{\times 3}{\_}\ 5\ \underset{\times 3}{\_}$

There must be at least one number other than 5, which we will denote N, between each 5. This eliminates 3 numbers N so there are only 3 more N to be placed

There are 3 possible positions before and after every 5, giving 15 possible positions.

There are $P(15,3) = \frac{15!}{12!}$ ways to order the remaining 3 numbers.

We conclude that there are

$5^6 \cdot \frac{15!}{12!}$ possible outcomes with exactly four 5's and no two 5's rolled in a row.

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3. [15 pts] Let $\mathbb{R}^+ = \{x \in \mathbb{R} | x > 0\}$ be the set of positive real numbers. Define a relation R on $\mathbb{R}^+$ by $(x, y) \in R$ if x/y is a rational number. Prove that R is an equivalence relation.

For any $x \in \mathbb{R}^+$, $x/x = 1 \in \mathbb{R}^+$ so $(x,x) \in R$. Therefore R is reflexive.

For any $x, y \in \mathbb{R}^+$, if $x/y \in \mathbb{R}^+$ then $y/x = 1/(y/x)$ is also a positive rational number so $(y,x) \in R$. Therefore R is symmetric.

For any $x, y, z \in \mathbb{R}^+$, if $x/y \in \mathbb{R}^+$ and $y/z \in \mathbb{R}^+$, then $x/z = (x/y)(y/z)$ is also a positive rational number as it is the product of two rational numbers. Therefore $(x,z) \in R$ so R is transitive.

Since R is reflexive, symmetric, and transitive, it is an equivalence relation.

Name: Justin Lau

4. [20 pts] Prove that for any positive integer n:

$$1(1!) + 2(2!) + 3(3!) + \cdots + n(n!) = (n+1)! - 1$$

[Hint: Use induction.]

We first verify the base case $n = 1$.

$1(1!) = (1+1)! - 1$

$1 = 2! - 1$

$1 = 2 - 1$

$1 = 1$

We now assume the statement is true for all k. We must show that the statement holds for $k+1$ as well.

$1(1!) + 2(2!) + \ldots + (k+1)(k+1)! = (k+2)! - 1$

We compute

$1(1!) + 2(2!) + \ldots + (k+1)(k+1)! = 1(1!) + 2(2!) + \ldots + k(k!) + (k+1)(k+1)!$

$= (k+1)! - 1 + (k+1)(k+1)!$

$= (k+1)!\,(1 + (k+1)) - 1$

$= (k+1)!\,(k+2) - 1$

$= (k+2)! - 1.$

We have proved the base case and inductive step.
Therefore, the statement is true for all positive integers n by mathematical induction.

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5. [20 pts] Let X, Y and Z be sets, and let $f : X \to Y$ and $g : Y \to Z$ be functions, and let $h = g \circ f$ be the composition of f and g. (That is, h is the function from X to Z defined by $h(x) = g(f(x))$.)

(a) [10 pts] Prove that if h is one-to-one, then f is one-to-one as well. [Hint: Assume that $f(x) = f(y)$, and try to use that to prove that $x = y$.]

We first assume that f(x) = f(y), we want to prove that x = y. Since h is one-to-one, there can only be one x ∈ X such that h(x) = z for some z ∈ Z. Therefore, if there were x and y such that x ≠ y and f(x) = f(y), we would have both x and y giving the same output h(x) = g(f(x)) = h(y) = g(f(y)) so h would not be one-to-one. Therefore if f(x) = f(y) then x = y so f is one-to-one if h is one-to-one.

(b) [10 pts] Give a counterexample to show that it is possible for h to be one-to-one while g is not one-to-one. That is, give examples of sets X, Y and Z and functions $f : X \to Y$ and $g : Y \to Z$ such that $h = g \circ f$ is one-to-one, but g is *not* one-to-one.

As a counterexample consider

X = {1, 2}, f : X → Y = {(1,1), (2,2)}

Y = {1, 2, 3} g : Y → Z = {(1,1), (2,2), (3,2)}

Z = {1,2}.

h = g∘f is one-to-one but g is not one-to-one.