Midterm 1 Solutions

Math 61, Lecture 2

Fall 2019

1. True/False

   (a) __**T**__ The sets $\{3, \clubsuit, \{1,2\}\}$ and $\{\{2,1,1\}, 3, \clubsuit, 3\}$ are equal.

   > **Solution:** These two sets have the same elements (notably, $\{1,2\} = \{2,1,1\}$).

   (b) __**T**__ There exists a surjective (i.e. onto) function:

   $$f : \mathcal{P}(\{1,2,3,4\}) \to \{A,B,C,D,E\} \times \{X,Y,Z\}$$

   where $\mathcal{P}(\{1,2,3,4\})$ is the power set of $\{1,2,3,4\}$.

   > **Solution:** $|\mathcal{P}(\{1,2,3,4\})| = 16 \geq 15 = |\{A,B,C,D,E\} \times \{X,Y,Z\}|$, so there is some surjection $\mathcal{P}(\{1,2,3,4\}) \to \{A,B,C,D,E\} \times \{X,Y,Z\}$. For example, we could enumerate the elements of $\mathcal{P}(\{1,2,3,4\})$ as $S_1, \ldots, S_{16}$ and enumerate the elements of $\{A,B,C,D,E\} \times \{X,Y,Z\}$ as $Y_1, \ldots, Y_{15}$, then define f by
   >
   > $$f(S_i) = \begin{cases} Y_i & : i \leq 15 \\ Y_{15} & : i = 16 \end{cases}$$

   (c) __**T**__ If $X = \{1,2,3,4,5,6\}$ then any injective (i.e. one-to-one) function $f : X \to X$ must also be surjective (i.e. onto).

   > **Solution:** Let R be the range of f. If f is not surjective, then $|R| < |X|$, so $f : X \to R$ is not injective.
   > Contrapositively, if f is injective, then f is surjective.

   (d) __**F**__ The relation R on $\mathbb{Z}$ defined by xRy if $x \neq y$ is antisymmetric.

   > **Solution:** $3R4$ and $4R3$ but $3 \neq 4$.

   (e) __**F**__ For any positive integers n and r with $r+1 \leq n$, $C(n,r) < C(n,r+1)$ (i.e. if X is an n-element set, there are always more $(r+1)$-combinations of X than r-combinations of X).

   > **Solution:** $r = 1, n = 2$ is a counterexample since $C(2,1) = 2 > 1 = C(2,2)$.

2. Compute the following quantities. You may leave your answers in terms of exponents and factorials, but do not leave your final answers in terms of $P(n,r)$ or $C(n,r)$ (so $4^{12}\frac{15!}{3!6!2!}$ would be an acceptable final answer, but $P(10,3)C(18,7)$ would not).

(a) The number of ways to form a 10 card hand from a standard 52 card deck(containing 13 clubs, 13 diamonds, 13 hearts and 13 spades) consisting of exactly 5 clubs, 3 diamonds, 2 hearts and no spades. (The order of the cards in this hand isirrelevant, only the set of 10 cards picked.)

Solution: First, choose 5 clubs: there are $\binom{13}{5}$ ways to do this. Then, choose 3 diamonds: there are $\binom{13}{3}$ ways to do this. Finally, choose 2 hearts: there are $\binom{13}{2}$ ways to do this. Our answer is thus

$$\binom{13}{5}\binom{13}{3}\binom{13}{2} = \frac{(13!)^3}{2!3!5!8!10!11!} = 28710396.$$

(b) The number of 6 letter strings that can be formed from the letters A, B, C, D, E (allowing repeats) which contain at least one A *and* at least one B.

Solution: We'll count the complement; let n be the number of 6-letter strings formed from the letters A, B, C, D, E which *do not* contain at least one A and at least one B. In other words, n is the number of 6-letter strings formed from the letters A, B, C, D, E which contain no A's or contain no B's (perhaps both). We can count this by inclusion-exclusion. The number of strings which contain no A's is just the number of 6-letter strings formed from the letters B, C, D, E, which is 4^6. Likewise, the number of strings which contain no B's is 4^6. The number of strings which contain no A's *and* no B's is just the number of 6-letter strings formed from the letters C, D, E, which is 3^6. By the inclusion-exclusion principle, $n = 4^6 + 4^6 - 3^6$. The total number of 6-letter strings formed from the letters A, B, C, D, E is 5^6, so our final answer is

$$5^6 - (4^6 + 4^6 - 3^6) = 8162.$$

(c) The number of permutations of the letters BOOKKEEPER (that is, strings of length 10 containing exactly 1 B, 2 O's, 2 K's, 3 E's, 1 P and 1 R) *such that the first E occurs before the first O.*

Solution: To make such a permutation, first choose how the 1 B, 2 K's, 1 P, and 1 R will be ordered. There are $\frac{5!}{2!}$ ways to do this. Next, choose a way to order the vowels such that the first E occurs before the first O. This means that the first of these five letters to appear must be an E, and the remaining 4 can appear in any order: there are thus $\binom{4}{2}$ ways to order the vowels. Now, we must "merge" our orderings, by deciding where among the E's and O's the other letters should go. There are 6 spots for the other letters to go (4 between the vowels and 2 at the ends), and we just need to decide how many consonants get placed in each of these potential spots. In other words, we need to find natural numbers $a_1, \ldots, a_6$ such that $a_1 + \cdots + a_6 = 5$; the number a_i tells us how many consonants to insert in the ith spot. By "stars and bars", there are $\binom{10}{5}$ ways to insert the consonants among the vowels. Our answer is therefore

$$\frac{5!}{2!}\binom{4}{2}\binom{10}{5} = \frac{4!10!}{(2!)^3 3!5!} = 90720.$$

3. In each problem below, the relation R is *NOT* an equivalence relation. In each problem, identify a specific property of equivalence relations which fails (either reflexivity, symmetry or transitivity), *and give a specific example to prove that it fails.*

   (a) $X = \mathcal{P}(\{1,2,3,4,5\})$, R is the relation on X defined by ARB is $A \subseteq B$ (i.e. A is a subset of B).

   Solution: This relation is not symmetric, e.g. $\varnothing R\{1\}$ but $\{1\}\not R\varnothing$.

   (b) R is the relation on $\mathbb{R}$ (the set of real numbers) defined by xRy if $|x-y| < 1$.

   Solution: This relation is not transitive, e.g. $0R0.5$ and $0.5R1$ but $0\not R1$.

   (c) $X = \{A | A \subseteq \{1,2,3,4,5,6\}$ and $|A| = 3\}$ is the set of *three element* subsets of $\{1,2,3,4,5,6\}$. R is the relation on X defined by ARB if $A \cap B \neq \varnothing$.

   Solution: This relation is not transitive, e.g. $\{1,2,3\}R\{1,2,4\}$ and $\{1,2,4\}R\{4,5,6\}$ but $\{1,2,3\}\not R\{4,5,6\}$.

4. Let X, Y and Z be sets, and let $f : X \to Y$ and $g : Y \to Z$ be functions, and let $h = g \circ f$ be the composition of f and g. (That is, h is the function from X to Z defined by $h(x) = g(f(x))$.)

   (a) Prove that if f and g are both onto, then h is onto as well.

   Solution: Let $z \in Z$ be arbitrary. By surjectivity of g, there is some $y \in Y$ such that $g(y) = z$. By surjectivity of f, there is some $x \in X$ such that $f(x) = y$. Now $h(x) = g(f(x)) = g(y) = z$, so z is an element of the range of h. Since z was arbitrary, this shows that h is surjective.

   (b) Prove that if f and g are both one-to-one, then h is one-to-one as well.

   Solution: Let $a, b \in X$ be arbitrary, and suppose that $h(a) = h(b)$; in other words, $g(f(a)) = g(f(b))$. Since g is injective, $f(a) = f(b)$. Since f is injective, $a = b$. Since a and B were arbitrary, this shows that h is injective.

5. Prove by induction that for any positive integer n,

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{(n-1)^2} + \frac{1}{n^2} \leq 2 - \frac{1}{n}.$$

Solution: We use induction on n.

Base Case When $n = 1$, then left-hand side and right-hand side are both equal to 1, so the inequality is verified.

Inductive Step Suppose that

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{(n-1)^2} + \frac{1}{n^2} \leq 2 - \frac{1}{n}$$

for some positive integer n. Then

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \frac{1}{(n+1)^2} \leq 2 - \frac{1}{n} + \frac{1}{(n+1)^2}. \tag{1}$$

Since

$$\frac{1}{n+1} \leq \frac{1}{n} = \frac{n+1}{n} - 1,$$

we see that

$$\frac{1}{(n+1)^2} \leq \frac{1}{n} - \frac{1}{n+1}.$$

In other words,

$$-\frac{1}{n} + \frac{1}{(n+1)^2} \leq -\frac{1}{n+1},$$

so

$$2 - \frac{1}{n} + \frac{1}{(n+1)^2} \leq 2 - \frac{1}{n+1}.$$

Using (1), we have transitively that

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \frac{1}{(n+1)^2} \leq 2 - \frac{1}{n+1},$$

as desired.