MIDTERM 1 (MATH 61)

MONDAY, OCTOBER 21ST

Name: [redacted]

ID: [redacted]

Circle your discussion section:

Tuesday	Thursday	
2A	2B	TA: Talon Stark
2C	2D	TA: Cameron Kissler
2E	2F	TA: Benjamin Spitz

This exam has 5 (double sided) pages, including the cover page, and a blank page at the end. Please make sure your exam includes each page. Please write your name on *each* page you submit. You will have 50 minutes to complete this exam. **You may not use a calculator**, or consult your textbook, class notes, or any other materials. If you need scratch paper or more space for your answers, please use the extra page at the end.

If there is any work on the blank pages which you would like to have graded, please indicate this CLEARLY on the page for the corresponding problem.

Show your work for these problems, don't just give an answer. If a question asks you to prove something, please write a complete proof. Unless otherwise stated, you may use any results proved in class or in the textbook, but please make it clear when you are doing so. Unless otherwise stated, you will *not* receive full credit for giving the correct answer with no explanation. You may still earn partial credit even if your final answer is incorrect.

Remember that you are bound by a conduct code, and that you may not look at anyone's paper or let anyone look at your paper.

Question	Points	Score
1	10	
2	35	
3	20	
4	15	
5	20	
Total:	100	

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1. [10 pts, 2 points each] Mark each of the following statements as either TRUE or FALSE. For this question you do not need to show any work beyond the final answer.

Be sure to read the questions carefully!

(a) The sets $\{3, \clubsuit, \{1,2\}\}$ and $\{\{2,1,1\}, 3, \clubsuit, 3\}$ are equal.	TRUE
(b) There exists a surjective (i.e. onto) function: $5 \times 4 = 20$ $2^4 = 16$ $$f : \mathcal{P}(\{1,2,3,4\}) \to \{A,B,C,D,E\} \times \{X,Y,Z\}$$ where $\mathcal{P}(\{1,2,3,4\})$ is the power set of $\{1,2,3,4\}$.	FALSE
(c) If $X = \{1,2,3,4,5,6\}$ then any injective (i.e. one-to-one) function $f : X \to X$ must also be surjective (i.e. onto).	TRUE
(d) The relation R on $\mathbb{Z}$ defined by xRy if $x \neq y$ is antisymmetric.	FALSE
(e) For any positive integers n and r with $r+1 \leq n$, $C(n,r) < C(n,r+1)$ (i.e. if X is an n-element set, there are always more $(r+1)$-combinations of X than r-combinations of X).	TRUE

$\binom{10}{3}$ $\binom{10}{4}$ $\frac{10!}{7!\,3!}$ $\frac{10!}{}$

$\frac{10}{0}$ $\frac{10}{1}$

$\frac{10!}{10!\,1!}$ $\frac{10!}{9!}$

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2. [35 pts] Compute the following quantities. You may leave your answers in terms of exponents and factorials, but do not leave your final answers in terms of $P(n,r)$ or $C(n,r)$ (so $4^{12}\frac{15!}{3!6!2!}$ would be an acceptable final answer, but $P(10,3)C(18,7)$ would not).

Show your work. It should be clear how you got your answers.

(a) [10 pts] The number of ways to form a 10 card hand from a standard 52 card deck (containing 13 clubs, 13 diamonds, 13 hearts and 13 spades) consisting of exactly 5 clubs, 3 diamonds, 2 hearts and no spades. (The order of the cards in this hand is irrelevant, only the set of 10 cards picked.)

$\boxed{\clubsuit}\ \boxed{\diamondsuit}\ \boxed{\heartsuit}$

$5 + 3 + 2 = 10$

$\frac{13!}{8!5!} \times \frac{13!}{10!3!} \times \frac{13!}{11!2!}$

[redacted]

(b) [10 pts] The number of 6 letter strings that can be formed from the letters A, B, C, D, E (allowing repeats) which contain at least one A *and* at least one B. [Hint: It may be easier to count the number of strings which don't satisfy this.]

Total # = 5^6

[redacted]

No A's = 4^5

No B's = 4^5

$5^6 - 4^5 - 4^5$

[redacted]

[redacted]

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(c) [15 pts] The number of permutations of the letters BOOKKEEPER (that is, strings of length 10 containing exactly 1 B, 2 O's, 2 K's, 3 E's, 1 P and 1 R) *such that the first E occurs before the first O.*

[So 'PR<u>E</u>B<u>O</u>KEEOK' would be one such permutation, but 'B<u>O</u>PK<u>E</u>REKOE' would not.]

[redacted]

Permutations of BooKKEEPER

_ E O _ _ _ _ _ _ _

$$= \frac{10!}{2!\,2!\,3!}$$

[redacted]

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3. [20 pts] In each problem below, the relation R is *NOT* an equivalence relation. In each problem, identify a specific property of equivalence relations which fails (either reflexivity, symmetry or transitivity), *and give a specific example to prove that it fails.*

(a) [5 pts] $X = \mathcal{P}(\{1,2,3,4,5\})$, R is the relation on X defined by ARB is $A \subseteq B$ (i.e. A is a subset of B).

Symmetry Fails

For example if $A = \{1,2\}$ and $B = \{1,2,3\}$, $A \subseteq B$ and ARB. But $B \nsubseteq A$ because $\{1,2,3\} \nsubseteq \{1,2\}$. So $B \not R A$ and this relation is NOT symmetric and thus not an equivalence relation.

[redacted]

(b) [7 pts] R is the relation on $\mathbb{R}$ (the set of real numbers) defined by xRy if $|x - y| < 1$.

Transitivity Fails

Let $x, y, z \in \mathbb{R}$ be such that $x = 0.5$, $y = 1$, and $z = 1.5$

$xRy \Rightarrow |0.5 - 1| < 1 \Rightarrow 0.5 < 1$ ✓

$yRz \Rightarrow |1 - 1.5| < 1 \Rightarrow 0.5 < 1$ ✓

However if we assume R is transitive, we get

$xRz \Rightarrow |0.5 - 1.5| \overset{?}{<} 1 \quad 1 \not< 1$

Thus $x, z \not\in R$ and R is NOT transitive.

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(c) [8 pts] $X = \{A | A \subseteq \{1,2,3,4,5,6\}$ and $|A| = 3\}$ is the set of *three element* subsets of $\{1,2,3,4,5,6\}$. R is the relation on X defined by ARB if $A \cap B \neq \varnothing$.

Transitivity Fails

Let's take sets $A, C \in X$ such that $A = \{1,2,3\}$ and $C = \{4,5,6\}$. Let $B = \{3,4,5\}$

If ARB, then we're saying $\{1,2,3\} \cap \{3,4,5\} \neq \phi$ which means $\{3\} \neq \phi$, which is true.

If BRC, then we're saying $\{3,4,5\} \cap \{4,5,6\} \neq \phi$ which means $\{4,5\} \neq \phi$, which is true.

If we assume R is transitive, then if ARB and BRC, then ARC. However we get

$\{1,2,3\} \cap \{4,5,6\} \stackrel{?}{\neq} \phi$

$\phi \neq \phi$, which is false.

Thus $A \not R C$ and our assumption was wrong so R is NOT transitive.

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4. [15 pts] Let X, Y and Z be sets, and let $f : X \to Y$ and $g : Y \to Z$ be functions, and let $h = g \circ f$ be the composition of f and g. (That is, h is the function from X to Z defined by $h(x) = g(f(x))$.)

(a) [7 pts] Prove that if f and g are both onto, then h is onto as well. [Hint: For any $z \in Z$, prove that there is some $x \in X$ with $h(x) = z$.]

If f is onto, then for every $y \in Y$, there is some $x \in X$ such that $f(x) = y$. For every $z \in Z$, there is also at some $y \in Y$ such that $g(y) = z$. Since we know $f(x)$ is onto, then EVERY element in Y is in its range. That means that every element in Y will be sent to $z \in Z$. Since $g(y)$ is onto, then we know every $z \in Z$ will be mapped to. Thus for any z, there is some $x \in X$ with $h(x) = z$

$[y] = X$

[redacted]

(b) [8 pts] Prove that if f and g are both one-to-one, then h is one-to-one as well. [Hint: Show that if $h(x_1) = h(x_2)$ for some $x_1, x_2 \in X$ then $x_1 = x_2$.]

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Knowing f is one-to-one

If $f(x_1) = y_1$, then $[y_1] = \{x_1\}$

If $f(x_2) = y_2$, then $[y_2] = \{x_2\}$

then if $y_1 = y_2$ then $[y_1] = [y_2]$

and $x_1 = x_2$.

[redacted]

If g is one to one, the same argument can be made that $g(y_1) = z_1 \to [z_1] = y_1$

$g(y_2) = z_2 \to [z_2] = \{y_2\}$

if $z_1 = z_2$, then $[z_1] = [z_2]$ and $y_1 = y_2$

From this we know $h(x_1) = g(f(x_1))$ and $h_2(x_2) = g(f(x_2))$ We know $f(x_1) = f(x_2)$ so $h(x_2) = g(f(x_1))$ Since $g(f(x_1)) = g(f(x_1))$, then we can say $h(x_1) = h(x_2)$ only when $x_1 = x_2$. Thus h is also one to one.

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5. [20 pts] Prove by induction that for any positive integer n,

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{(n-1)^2}+\frac{1}{n^2}\le 2-\frac{1}{n}.$$

Base: $\frac{1}{1^2} \le 2 - \frac{1}{n}$

$1 \le 2 - 1$

$1 \le 1$ ✓

Inductive step: Assume $\frac{1}{1^2} + \dots + \frac{1}{(n-1)^2} + \frac{1}{n^2} \le 2 - \frac{1}{n}$ is true,

then $\underbrace{\frac{1}{1^2} + \dots + \frac{1}{(n-1)^2} + \frac{1}{n^2}} + \frac{1}{(n+1)^2} \le 2 - \frac{1}{n+1}$

[redacted]

$2 - \frac{1}{n}\frac{(n+1)^2}{(n+1)^2} \frac{n}{n(n+1)^2} \le 2 - \frac{1}{n+1}$

$\frac{(n-(n+1)^2}{n(n+1)^2} \le -\frac{1}{n+1}$

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