19F-MATH61-2 Midterm 1

TOTAL POINTS

93 / 100

QUESTION 1

True/False 10 pts

1.1 a 2 / 2

✓ - 0 pts True - Correct

1.2 b 2 / 2

✓ - 0 pts True - Correct

1.3 c 2 / 2

✓ - 0 pts True -Correct

1.4 d 2 / 2

✓ - 0 pts False - Correct

1.5 e 2 / 2

✓ - 0 pts False - Correct

QUESTION 2

Counting 35 pts

2.1 10-card Hands 10 / 10

✓ - 0 pts Correct

2.2 6-letter Strings 10 / 10

✓ - 0 pts Correct

2.3 Permutations of Bookkeeper 15 / 15

✓ - 0 pts Correct

QUESTION 3

Not Equivalence Relations 20 pts

3.1 Subset 5 / 5

✓ - 0 pts Correct

3.2 Distance <1 7 / 7

✓ - 0 pts Correct

3.3 Nontrivial Intersection 8 / 8

✓ - 0 pts Correct

QUESTION 4

Function Composition 15 pts

4.1 Surjective 7 / 7

✓ - 0 pts Correct

4.2 Injective 1 / 8

✓ - 7 pts Proved that x_1=x_2 implies h(x_1) = h(x_2) (or something similar) instead of correct implication.

QUESTION 5

5 Induction 20 / 20

+ 4 pts Base Case
+ 6 pts Correct goal for inductive step
+ 5 pts Almost correctly executed inductive step
+ 10 pts Correctly executed inductive step

✓ + 20 pts Correct

+ 0 pts Click here to replace this description.

Midterm 1 (Math 61)

Monday, October 21st

Nam ______________________________

ID:_ ______________________________

Circle your discussion section:

Tuesday	Thursday	
2A	2B	TA: Talon Stark
2C	2D	TA: Cameron Kissler
2E	2F	TA: Benjamin Spitz

This exam has 5 (double sided) pages, including the cover page, and a blank page at the end. Please make sure your exam includes each page. Please write your name on *each* page you submit. You will have 50 minutes to complete this exam. **You may not use a calculator**, or consult your textbook, class notes, or any other materials. If you need scratch paper or more space for your answers, please use the extra page at the end.

If there is any work on the blank pages which you would like to have graded, please indicate this CLEARLY on the page for the corresponding problem.

Show your work for these problems, don't just give an answer. If a question asks you to prove something, please write a complete proof. Unless otherwise stated, you may use any results proved in class or in the textbook, but please make it clear when you are doing so. Unless otherwise stated, you will *not* receive full credit for giving the correct answer with no explanation. You may still earn partial credit even if your final answer is incorrect.

Remember that you are bound by a conduct code, and that you may not look at anyone's paper or let anyone look at your paper.

Question	Points	Score
1	10	
2	35	
3	20	
4	15	
5	20	
Total:	100	

Name:____________________

1. [10 pts, 2 points each] Mark each of the following statements as either TRUE or FALSE. For this question you do not need to show any work beyond the final answer.

Be sure to read the questions carefully!

$\{1,2\}$

(a) The sets $\{3, \clubsuit, \{1,2\}\}$ and $\{\{2,1,1\}, 3, \clubsuit, 3\}$ are equal.	T
(b) There exists a surjective (i.e. onto) function: $2^4 = 16$ $\quad 5 \quad 3$ $$f : \mathcal{P}(\{1,2,3,4\}) \to \{A,B,C,D,E\} \times \{X,Y,Z\} \to 15$$ where $\mathcal{P}(\{1,2,3,4\})$ is the power set of $\{1,2,3,4\}$.	T
(c) If $X = \{1,2,3,4,5,6\}$ then any injective (i.e. one-to-one) function $f : X \to X$ must also be surjective (i.e. onto).	T
(d) The relation R on $\mathbb{Z}$ defined by xRy if $x \neq y$ is antisymmetric. if xRy and yRx then $x = y$; $x \neq y$ $\quad y \neq x$	F
(e) For any positive integers n and r with $r+1 \leq n$, $C(n,r) < C(n,r+1)$ (i.e. if X is an n-element set, there are always more $(r+1)$-combinations of X than r-combinations of X).	F

1	1
2	2
3	3
4	4
5	5
6	6

$\binom{n}{r} \quad \binom{n}{r+1}$

$\binom{3}{1} = \binom{3}{2}$

Name:____________________________

2. [35 pts] Compute the following quantities. You may leave your answers in terms of exponents and factorials, but do not leave your final answers in terms of $P(n,r)$ or $C(n,r)$ (so $4^{12}\frac{15!}{3!6!2!}$ would be an acceptable final answer, but $P(10,3)C(18,7)$ would not).

Show your work. It should be clear how you got your answers.

(a) [10 pts] The number of ways to form a 10 card hand from a standard 52 card deck (containing 13 clubs, 13 diamonds, 13 hearts and 13 spades) consisting of exactly 5 clubs, 3 diamonds, 2 hearts and no spades. (The order of the cards in this hand is irrelevant, only the set of 10 cards picked.)

There are 13 clubs, so $\binom{13}{5}$ ways to pick 5

13 diamonds, so $\binom{13}{3}$ ways to pick 3

13 hearts, so $\binom{13}{2}$ ways to pick 2

$\Big\}$ total = 10 cards

Altogether, that is $\binom{13}{5}\binom{13}{3}\binom{13}{2}$

$$\boxed{\left(\frac{13!}{5!\,8!}\right)\left(\frac{13!}{3!\,10!}\right)\left(\frac{13!}{2!\,11!}\right)}$$

(b) [10 pts] The number of 6 letter strings that can be formed from the letters A, B, C, D, E (allowing repeats) which contain at least one A *and* at least one B. [Hint: It may be easier to count the number of strings which don't satisfy this.]

Strings that don't satisfy: no B, no A, no B & A

total # of 6 letter strings : 5^6

w/ no B : 4^6

w/no A = 4^6

w/ no B & A = 3^6

There's overlap when counting strings w/ no B & strings w/ no A,

so total # that don't satisfy is: $2(4^6) - 3^6$

so strings which contain at least one A and at least one B:

$$\boxed{5^6 - (2(4^6) - 3^6)}$$

Name:________________________________

(c) [15 pts] The number of permutations of the letters BOOKKEEPER (that is, strings of length 10 containing exactly 1 B, 2 O's, 2 K's, 3 E's, 1 P and 1 R) *such that the first E occurs before the first O.*

[So 'PRE̲BO̲KEEOK' would be one such permutation, but 'BO̲PKE̲REKOE' would not.]

ways to order E's and O's = 5!

$3 \times 4!$ → # of ways first E comes before 1st O

so only $\frac{3 \times 4!}{5!} = \frac{3}{5}$ of orderings of E's and O's are valid

ways to order BOOKKEEPER = $\frac{10!}{2!\,2!\,3!}$ (b/c there's repeats)

$$\boxed{\frac{3}{5}\left(\frac{10!}{2!\,2!\,3!}\right)}$$

Name:________________________________

3. [20 pts] In each problem below, the relation R is *NOT* an equivalence relation. In each problem, identify a specific property of equivalence relations which fails (either reflexivity, symmetry or transitivity), *and give a specific example to prove that it fails.*

(a) [5 pts] $X = \mathcal{P}(\{1,2,3,4,5\})$, R is the relation on X defined by ARB is $A \subseteq B$ (i.e. A is a subset of B).

R is not an equivalence relation because it's not symmetric.

If ARB $(A \subseteq B)$, it is not necessarily true that BRA $(B \subseteq A)$.

As an example, if $A = \phi$, $B = \{1,2\}$

then $A \subseteq B$ because the empty is a subset of any set,

but $B \not\subseteq A$ because $\{1,2\}$ is not a subset of the empty set.

Therefore, R is not symmetric and not an equivalence relation.

(b) [7 pts] R is the relation on $\mathbb{R}$ (the set of real numbers) defined by xRy if $|x - y| < 1$.

R is not transitive. If xRy, or $|x-y| < 1$, and yRz, or $|y-z| < 1$,

it does not necessarily mean xRz, or $|x-z| < 1$.

For example, if $x = -\frac{1}{2}$, $y = -1$, $z = -\frac{3}{2}$

then $|x-y| = |-\frac{1}{2}+1| < 1$ and $|y-z| = |-1+\frac{3}{2}| < 1$,

but $|x-z| = |-\frac{1}{2}+\frac{3}{2}| \nless 1$, so R is not transitive.

Therefore, R is not an equivalence relation.

Name:________________________________

(c) [8 pts] $X = \{A | A \subseteq \{1,2,3,4,5,6\}$ and $|A| = 3\}$ is the set of *three element* subsets of $\{1,2,3,4,5,6\}$. R is the relation on X defined by ARB if $A \cap B \neq \varnothing$.

R is not transitive b/c if $A \cap B \neq \emptyset$ and $B \cap C \neq \emptyset$ it does not mean that $A \cap C \neq \emptyset$.

Forexample, if $A = \{1,2,3\}$ and $B = \{3,4,5\}$ and $C = \{4,5,6\}$,

$A \cap B = \{3\}$, $B \cap C = \{4,5\}$, but $A \cap C = \emptyset$

so R is not transitive, and therefore not an equivalence relation.

Name:____________________________

4. [15 pts] Let X, Y and Z be sets, and let $f : X \to Y$ and $g : Y \to Z$ be functions, and let $h = g \circ f$ be the composition of f and g. (That is, h is the function from X to Z defined by $h(x) = g(f(x))$.)

(a) [7 pts] Prove that if f and g are both onto, then h is onto as well. [Hint: For any $z \in Z$, prove that there is some $x \in X$ with $h(x) = z$.]

If f is onto, then for any $y \in Y$, there is some $x \in X$ such that $f(x) = y$. If g is onto, then for any $z \in Z$, there is some $y \in Y$ such that $g(y) = z$. Since there is always an $x \in X$ for $f(x) \in Y$, and there is always an $f(x) \in Y$ such that $g(f(x)) = z$ for all $z \in Z$, that means that for any $z \in Z$, there is an $x \in X$ such that $h(x) = z$.
So, h is also onto.

(b) [8 pts] Prove that if f and g are both one-to-one, then h is one-to-one as well. [Hint: Show that if $h(x_1) = h(x_2)$ for some $x_1, x_2 \in X$ then $x_1 = x_2$.]

Since f and g are both one-to-one, that means for $x_1, x_2 \in X$, if $x_1 = x_2$, then $f(x_1) = f(x_2)$
and for any $f(x_1), f(x_2) \in Y$, if $f(x_1) = f(x_2)$, then
$g(f(x_1)) = g(f(x_2))$
which means that $h(x_1) = h(x_2)$ for $x_1, x_2 \in X$ if
$x_1 = x_2$. Therefore, h is one-to-one.

Name:______________________________

5. [20 pts] Prove by induction that for any positive integer n,

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{(n-1)^2}+\frac{1}{n^2}\le 2-\frac{1}{n}.$$

Base Case : $n=1$ (1st positive integer)

$\frac{1}{1^2} \le 2-\frac{1}{1} \rightarrow 1 \le 1$, which is true

Let's assume $\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{(n-1)^2}+\frac{1}{n^2}\le 2-\frac{1}{n}$ is true.

Then, for $n+1$: $\underbrace{\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{n^2}}+\frac{1}{(n+1)^2}\le 2-\frac{1}{n+1}$

$$2-\frac{1}{n}+\frac{1}{(n+1)^2}\le 2-\frac{1}{n+1}$$

$$2-\left(\frac{1}{n}-\frac{1}{(n+1)^2}\right)\le 2-\frac{1}{n+1}$$

$$2-\frac{n(n+1)^2-n}{(n+1)^2 n}\le 2-\frac{n+1}{(n+1)^2}$$

$$2-\frac{(n+1)^2-1}{(n+1)^2}\le 2-\frac{n+1}{(n+1)^2} \checkmark$$

b/c $\frac{(n+1)^2-1}{(n+1)^2}\ge\frac{n+1}{(n+1)^2}$

Therefore, since this is true for the base case $(n=1)$ and it is true for $n+1$ if we assume it is true for n, by induction,

$\frac{1}{1^2}+\cdots+\frac{1}{n^2}\le 2-\frac{1}{n}$ for any positive integer n.

Name:______________________________

This page is intentionally blank. You may use it if you need more room for your solutions.

Name:________________________________

This page is intentionally blank. You may use it if you need more room for your solutions.