Midterm 1 (Math 61)

Monday, October 21st

Name: Miles Kang

ID: [redacted]

Circle your discussion section:

Tuesday	Thursday	
2A	2B	TA: Talon Stark
2C	2D	TA: Cameron Kissler
2E	2F	TA: Benjamin Spitz

This exam has 5 (double sided) pages, including the cover page, and a blank page at the end. Please make sure your exam includes each page. Please write your name on *each* page you submit. You will have 50 minutes to complete this exam. **You may not use a calculator**, or consult your textbook, class notes, or any other materials. If you need scratch paper or more space for your answers, please use the extra page at the end.

If there is any work on the blank pages which you would like to have graded, please indicate this CLEARLY on the page for the corresponding problem.

Show your work for these problems, don't just give an answer. If a question asks you to prove something, please write a complete proof. Unless otherwise stated, you may use any results proved in class or in the textbook, but please make it clear when you are doing so. Unless otherwise stated, you will *not* receive full credit for giving the correct answer with no explanation. You may still earn partial credit even if your final answer is incorrect.

Remember that you are bound by a conduct code, and that you may not look at anyone's paper or let anyone look at your paper.

Question	Points	Score
1	10	
2	35	
3	20	
4	15	
5	20	
Total:	100	

Name: Miles Kang

1. [10 pts, 2 points each] Mark each of the following statements as either TRUE or FALSE. For this question you do not need to show any work beyond the final answer.

Be sure to read the questions carefully!

(a) The sets $\{3, \clubsuit, \{1,2\}\}$ and $\{\{2,1,1\}, 3, \clubsuit, 3\}$ are equal.	F
(b) There exists a surjective (i.e. onto) function: $2^4 = 16$ $>$ $3 \cdot 5 = 15$ $$f : \mathcal{P}(\{1,2,3,4\}) \to \{A,B,C,D,E\} \times \{X,Y,Z\}$$ where $\mathcal{P}(\{1,2,3,4\})$ is the power set of $\{1,2,3,4\}$.	T
(c) If $X = \{1,2,3,4,5,6\}$ then any injective (i.e. one-to-one) function $f : X \to X$ must also be surjective (i.e. onto).	T
(d) The relation R on $\mathbb{Z}$ defined by xRy if $x \neq y$ is antisymmetric.	F
(e) For any positive integers n and r with $r + 1 \leq n$, $C(n,r) < C(n,r+1)$ (i.e. if X is an n-element set, there are always more $(r+1)$-combinations of X than r-combinations of X).	F

Name:________________________________

2. [35 pts] Compute the following quantities. You may leave your answers in terms of exponents and factorials, but do not leave your final answers in terms of $P(n,r)$ or $C(n,r)$ (so $4^{12}\frac{15!}{3!6!2!}$ would be an acceptable final answer, but $P(10,3)C(18,7)$ would not).

Show your work. It should be clear how you got your answers.

(a) [10 pts] The number of ways to form a 10 card hand from a standard 52 card deck (containing 13 clubs, 13 diamonds, 13 hearts and 13 spades) consisting of exactly 5 clubs, 3 diamonds, 2 hearts and no spades. (The order of the cards in this hand is irrelevant, only the set of 10 cards picked.)

$$\binom{13}{5}\cdot\binom{13}{3}\cdot\binom{13}{2}\cdot\binom{13}{0}$$

$$= \boxed{\frac{13!}{5!\,8!}\cdot\frac{13!}{3!\,10!}\cdot\frac{13!}{2!\,11!}}$$

(b) [10 pts] The number of 6 letter strings that can be formed from the letters A, B, C, D, E (allowing repeats) which contain at least one A *and* at least one B. [Hint: It may be easier to count the number of strings which don't satisfy this.]

_ _ _ _ _ _

Strings w/ neither A nor B: 3^6

w/ at least one A: 4^6-3^6

w/ at least one B: 4^6-3^6

Total: 5^6

$\hookrightarrow$ $5^6-3^6-2(4^6-3^6)$

$= \boxed{5^6-2(4^6)+3^6}$

Name:________________________________

(c) [15 pts] The number of permutations of the letters BOOKKEEPER (that is, strings of length 10 containing exactly 1 B, 2 O's, 2 K's, 3 E's, 1 P and 1 R) *such that the first E occurs before the first O.*

[So 'PRE̲BO̲KEEOK' would be one such permutation, but 'BO̲PKE̲REKOE' would not.]

1) Rearrange the vowels such that first E occurs before first O:

$\underline{E}$ _ _ _ _

↑

must be E

↳ $\frac{4!}{2!2!}$

2) Ways to rearrange consonants: $\frac{5!}{2!}$

3) o o o o o o o o o o

Ways to pick 5 to be vowels:

$\binom{10}{5}$

↳ $\binom{10}{5} \times \frac{5!}{2!} \times \frac{4!}{2!2!}$

$= \frac{10!}{5!\cancel{5!}} \times \frac{\cancel{5!}}{2!} \times \frac{4!}{2!2!}$

$= \frac{10!\,4!}{5! \cdot 2^3}$

Name:____________________________

3. [20 pts] In each problem below, the relation R is *NOT* an equivalence relation. In each problem, identify a specific property of equivalence relations which fails (either reflexivity, symmetry or transitivity), *and give a specific example to prove that it fails.*

(a) [5 pts] $X = \mathcal{P}(\{1, 2, 3, 4, 5\})$, R is the relation on X defined by ARB is $A \subseteq B$ (i.e. A is a subset of B).

reflexivity fails.

example: A is $\{1\}$

B is $\{1, 2\}$

$A \subseteq B$ but $B \nsubseteq A$.

(b) [7 pts] R is the relation on $\mathbb{R}$ (the set of real numbers) defined by xRy if $|x - y| < 1$.

transitivity fails.

example: Let $x = 1$, $y = 0.2$, $z = -0.2$.

xRy because $|1 - 0.2| = .8 < 1$

yRz because $|0.2 - (-0.2)| = .4 < 1$

However, $|x - z| = |1 - (-0.2)| = 1.2 \nless 1$.

Name:________________________________

(c) [8 pts] $X = \{A | A \subseteq \{1, 2, 3, 4, 5, 6\}$ and $|A| = 3\}$ is the set of *three element* subsets of $\{1, 2, 3, 4, 5, 6\}$. R is the relation on X defined by ARB if $A \cap B \neq \varnothing$.

Transitivity fails.

example: Let A be $\{1,2,3\}$, B be $\{3,4,5\}$, C be $\{4,5,6\}$.

ARB because $A \cap B = \{3\} \neq \emptyset$.

BRC because $B \cap C = \{4,5\} \neq \emptyset$.

However, $A \cap C = \emptyset$.

So, R is not transitive.

Name:______________________________

4. [15 pts] Let X, Y and Z be sets, and let $f : X \to Y$ and $g : Y \to Z$ be functions, and let $h = g \circ f$ be the composition of f and g. (That is, h is the function from X to Z defined by $h(x) = g(f(x))$.)

(a) [7 pts] Prove that if f and g are both onto, then h is onto as well. [Hint: For any $z \in Z$, prove that there is some $x \in X$ with $h(x) = z$.]

$h(x) = g(f(x))$.

Let $f(x) = y$ and $g(y) = z$.

Since f and g are onto, there is some $y \in Y$ with $f(x) = y$,

and some $z \in Z$ with $g(y) = z$.

Since $h(x) = g(f(x)) = g(y) = z$, there is always some $x \in X$ with $h(x) = z$.

So, h is onto if f and g are both onto. ✓

(b) [8 pts] Prove that if f and g are both one-to-one, then h is one-to-one as well. [Hint: Show that if $h(x_1) = h(x_2)$ for some $x_1, x_2 \in X$ then $x_1 = x_2$.]

$$\begin{cases} h(x_1) = g(f(x_1)) \\ h(x_2) = g(f(x_2)) \\ h(x_1) = h(x_2) \to g(f(x_1)) = g(f(x_2)). \end{cases}$$

Since g is one-to-one, $f(x_1) = f(x_2)$.

Since f is one-to-one, $x_1 = x_2$.

Therefore, if $h(x_1) = h(x_2)$, $x_1 = x_2$.

So, h is one-to-one if f and g are both one-to-one. ✓

Name:______________________________

5. [20 pts] Prove by induction that for any positive integer n,

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{(n-1)^2} + \frac{1}{n^2} \leq 2 - \frac{1}{n}.$$

Base case: $n=1$

$$\frac{1}{1^2} \leq 2 - \frac{1}{1}$$

$$1 \leq 1 \checkmark$$

Inductive step: Suppose the statement is true for $n=k$, where $k \in \mathbb{Z}^+$. Then it must also be true for $n=k+1$.

$$\frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{k^2} \leq 2 - \frac{1}{k}.$$

$$\underbrace{\frac{1}{1^2} + \frac{1}{2^2} + \dots}_{} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k+1}.$$

$$2 - \frac{1}{k} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k+1}$$

$$2 - \left(\frac{1}{k} - \frac{1}{(k+1)^2}\right) \leq 2 - \frac{1}{k+1}$$

$$\cancel{2} - \left(\frac{(k+1)^2 - k}{k(k+1)^2}\right) \leq \cancel{2} - \frac{1}{k+1}$$

$$-\left(\frac{k^2+k+1}{k(k+1)^2}\right) \leq \frac{-1}{k+1}$$

$$\left(\frac{k^2+k+1}{k(k+1)^2}\right) \geq \frac{1}{k+1}$$

$$\frac{k^2+k+1}{k(k+1)} \geq 1 \quad \leftarrow \text{(safe to do because } k>0)$$

$$\frac{k^2+k+1}{k^2+k} \geq 1$$

$$1 + \frac{1}{k^2+k} \geq 1. \quad \rightarrow \frac{1}{k^2+k} \geq 0.$$

Since k is a positive integer, this statement is true. ✓

Since the statement holds for $n=k+1$, it must hold for all positive integers n. ✓

Name:________________________________

This page is intentionally blank. You may use it if you need more room for your solutions.

Name:______________________________

This page is intentionally blank. You may use it if you need more room for your solutions.