19F-MATH61-2 Midterm 1

MAANEK SEHGAL

TOTAL POINTS

89 / 100

QUESTION 1

True/False 10 pts

1.1 a **2 / 2**

✓ **- 0 pts True - Correct**

1.2 b **2 / 2**

✓ **- 0 pts True - Correct**

1.3 c **2 / 2**

✓ **- 0 pts True -Correct**

1.4 d **2 / 2**

✓ **- 0 pts False - Correct**

1.5 e **2 / 2**

✓ **- 0 pts False - Correct**

QUESTION 2

Counting 35 pts

2.1 10-card Hands **10 / 10**

✓ **- 0 pts Correct**

2.2 6-letter Strings **10 / 10**

✓ **- 0 pts Correct**

2.3 Permutations of Bookkeeper **15 / 15**

✓ **- 0 pts Correct**

QUESTION 3

Not Equivalence Relations 20 pts

3.1 Subset **5 / 5**

✓ **- 0 pts Correct**

3.2 Distance <1 **7 / 7**

✓ **- 0 pts Correct**

3.3 Nontrivial Intersection **4 / 8**

- 4 Point adjustment

Misunderstood definition of X

QUESTION 4

Function Composition 15 pts

4.1 Surjective **6 / 7**

✓ **- 0 pts Correct**

- 1 Point adjustment

This could use some clearer explanation for why it actually follows that every z will be in the range of h. It's better to use precise terms like "for every z in Z there will be a y in Y with g(y) = z, then there will be a x in X with f(x) = y, so h(x) = z" instead of vague terms like saying the "whole domain is "accessible" in g o f."

4.2 Injective **8 / 8**

✓ **- 0 pts Correct**

QUESTION 5

5 Induction **14 / 20**

✓ **+ 4 pts Base Case**

✓ **+ 6 pts Correct goal for inductive step**

\+ **5 pts** Almost correctly executed inductive step

\+ **10 pts** Correctly executed inductive step

\+ **20 pts** Correct

\+ **0 pts** Click here to replace this description.

+ 4 Point adjustment

Midterm 1 (Math 61)

Monday, October 21st

Name: MAANEK SEHGAL

ID: 505165291

Circle your discussion section:

Tuesday	Thursday	
2A	2B	TA: Talon Stark
2C	2D	TA: Cameron Kissler
2E	2F	TA: Benjamin Spitz

This exam has 5 (double sided) pages, including the cover page, and a blank page at the end. Please make sure your exam includes each page. Please write your name on *each* page you submit. You will have 50 minutes to complete this exam. **You may not use a calculator**, or consult your textbook, class notes, or any other materials. If you need scratch paper or more space for your answers, please use the extra page at the end.

If there is any work on the blank pages which you would like to have graded, please indicate this CLEARLY on the page for the corresponding problem.

Show your work for these problems, don't just give an answer. If a question asks you to prove something, please write a complete proof. Unless otherwise stated, you may use any results proved in class or in the textbook, but please make it clear when you are doing so. Unless otherwise stated, you will *not* receive full credit for giving the correct answer with no explanation. You may still earn partial credit even if your final answer is incorrect.

Remember that you are bound by a conduct code, and that you may not look at anyone's paper or let anyone look at your paper.

Question	Points	Score
1	10	
2	35	
3	20	
4	15	
5	20	
Total:	100	

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1. [10 pts, 2 points each] Mark each of the following statements as either TRUE or FALSE. For this question you do not need to show any work beyond the final answer.

Be sure to read the questions carefully!

(a) The sets $\{3, \clubsuit, \{1,2\}\}$ and $\{\{2,1,1\}, 3, \clubsuit, 3\}$ are equal.	True
(b) There exists a surjective (i.e. onto) function: 5 3 15 $$f : \mathcal{P}(\{1,2,3,4\}) \to \{A,B,C,D,E\} \times \{X,Y,Z\}$$ where $\mathcal{P}(\{1,2,3,4\})$ is the power set of $\{1,2,3,4\}$. $4! = 4 \cdot 3 \cdot 2 = 24$	True
$\lvert X \rvert = \lvert Y \rvert \to$ (c) If $X = \{1,2,3,4,5,6\}$ then any injective (i.e. one-to-one) function $f : X \to X$ must also be surjective (i.e. onto).	True
(d) The relation R on $\mathbb{Z}$ defined by xRy if $x \neq y$ is antisymmetric. $3R4$ $4R3$	False
(e) For any positive integers n and r with $r+1 \leq n$, $C(n,r) < C(n,r+1)$ (i.e. if X is an n-element set, there are always more $(r+1)$-combinations of X than r-combinations of X).	False

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2. [35 pts] Compute the following quantities. You may leave your answers in terms of exponents and factorials, but do not leave your final answers in terms of $P(n,r)$ or $C(n,r)$ (so $4^{12}\frac{15!}{3!6!2!}$ would be an acceptable final answer, but $P(10,3)C(18,7)$ would not).

Show your work. It should be clear how you got your answers.

(a) [10 pts] The number of ways to form a 10 card hand from a standard 52 card deck (containing 13 clubs, 13 diamonds, 13 hearts and 13 spades) consisting of exactly 5 clubs, 3 diamonds, 2 hearts and no spades. (The order of the cards in this hand is irrelevant, only the set of 10 cards picked.)

clubs diamonds hearts

$$\binom{13}{5}\cdot\binom{13}{3}\cdot\binom{13}{2}$$

$$= \boxed{\frac{13!}{5!\,8!}\cdot\frac{13!}{3!\,10!}\cdot\frac{13!}{11!\,2!}}$$

(b) [10 pts] The number of 6 letter strings that can be formed from the letters A, B, C, D, E (allowing repeats) which contain at least one A *and* at least one B. [Hint: It may be easier to count the number of strings which don't satisfy this.]

no A's or no B's minus (no A's and no B's)

4^6 (4 letters) 4^6 (4 letters) 3^6 (3 letters)

total # of strings

$$\boxed{5^6 - 2(4^6) + 3^6}$$

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(c) [15 pts] The number of permutations of the letters BOOKKEEPER (that is, strings of length 10 containing exactly 1 B, 2 O's, 2 K's, 3 E's, 1 P and 1 R) *such that the first E occurs before the first O.*
[So 'PREBOKEEOK' would be one such permutation, but 'BOPKEREKOE' would not.]

$$\underset{\text{fixed}}{\underline{E}}\ \overbrace{E\ E\ O\ O}\ \rightarrow \frac{4!}{2!\,2!}$$

BKKPR

$$\frac{\#\text{ of good EO orderings}}{\text{total }\#\text{ of EO orderings}} \cdot \text{total orderings}$$

$$\cancel{\frac{4!}{2!\,2!} \cdot \frac{5!}{2!}}$$

$$\frac{\frac{4!}{2!\,\cancel{2!}}}{\frac{5!}{3!\,\cancel{2!}}} \cdot \frac{10!}{3!\,2!\,2!}$$

$$\frac{4!\,\cancel{3!}}{5!\,2!} \cdot \frac{10!}{\cancel{3!}\,2!\,2!}$$

$\underset{1}{\_}\ E\ \underset{2}{\_}\ E\ \underset{3}{\_}\ E\ \underset{4}{\_}\ O\ \underset{5}{\_}\ O\ \underset{6}{\_}$

5 letters left

$$\boxed{\frac{4!}{2!\,2!} \cdot \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{2!}}$$

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3. [20 pts] In each problem below, the relation R is *NOT* an equivalence relation. In each problem, identify a specific property of equivalence relations which fails (either reflexivity, symmetry or transitivity), *and give a specific example to prove that it fails.*

(a) [5 pts] $X = \mathcal{P}(\{1, 2, 3, 4, 5\})$, R is the relation on X defined by ARB is $A \subseteq B$ (i.e. A is a subset of B).

reflexive: every set is a subset of itself ✓

transitive: $A \subseteq B$ and $B \subseteq C \rightarrow A \subseteq C$ ✓

Symmetric: no.

let $A = \{1, 2, 3\}$ and let $B = \{1, 2\}$. $A \in X$ and $B \in X$. $B \subseteq A$ so BRA. HOWEVER, $A \not\subseteq B$ so $(A, B) \notin R$. Therefore, R is not symmetric.

(b) [7 pts] R is the relation on $\mathbb{R}$ (the set of real numbers) defined by xRy if $|x - y| < 1$.

let $x = 1$, $y = 0.1$, $z = 1.9$

$|y - x| = |0.1 - 1| = 0.9 < 1$

Therefore, yRx

$|x - z| = |1 - 1.9| = 0.9 < 1$ therefore xRz

$|y - z| = |0.1 - 1.9| = 1.8 \not< 1$

There fore, $(y, z) \notin R$ even though yRx and xRz. Thus, R is not transitive.

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(c) [8 pts] $X = \{A | A \subseteq \{1,2,3,4,5,6\} \text{ and } |A| = 3\}$ is the set of *three element* subsets of $\{1,2,3,4,5,6\}$. R is the relation on X defined by ARB if $A \cap B \neq \varnothing$.

let $A = \varnothing$ (a valid subset of $\{1,2,3,4,5,6\}$)

~~let $B = \varnothing$ (a valid subset of $\{1,2,3,4,5,6\}$)~~

$A \cap A = \varnothing$, therefore $(A, A) \notin R$.

However, $A \in X$. Therefore, R is not reflexive because the statement $(B, B) \in R \ \forall B \in X$ is not true.

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4. [15 pts] Let X, Y and Z be sets, and let $f : X \to Y$ and $g : Y \to Z$ be functions, and let $h = g \circ f$ be the composition of f and g. (That is, h is the function from X to Z defined by $h(x) = g(f(x))$.)

(a) [7 pts] Prove that if f and g are both onto, then h is onto as well. [Hint: For any $z \in Z$, prove that there is some $x \in X$ with $h(x) = z$.]

Onto: for any element in codomain, there is at least one element in the domain that "maps" to it.
If f is onto, then every $y \in Y$ will be an input for g in $g \circ f$. If g is onto, then, since its whole domain is "accessible" in $g \circ f$, every $z \in Z$ will be in the range of $h(x)$. If every $z \in Z$ is in the range, then $h(x)$ is onto.

~~(b)~~ [8 pts] Prove that if f and g are both one-to-one, then h is one-to-one as well. [Hint: Show that if $h(x_1) = h(x_2)$ for some $x_1, x_2 \in X$ then $x_1 = x_2$.]

Assume $h(x_1) = h(x_2)$ for some $x_1, x_2 \in X$. Then, $g(f(x_1)) = g(f(x_2))$. Since g is injective, $f(x_1)$ must equal $f(x_2)$. Since f is injective, $x_1 = x_2$. Therefore, if $h(x_1) = h(x_2)$, $x_1 = x_2$. Thus, h is one-to-one.

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5. [20 pts] Prove by induction that for any positive integer n, $n \geq 1$

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{(n-1)^2}+\frac{1}{n^2} \leq 2-\frac{1}{n}.$$ Induct on n

Base Case:
$n=1$

$$\frac{1}{1^2} \leq 2-\frac{1}{1} \rightarrow 1 \leq 2-1 \rightarrow 1 \leq 1 \checkmark$$

Inductive Step:

Assume $\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{n^2} \leq 2-\frac{1}{n}$

Goal: $\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{n^2}+\frac{1}{(n+1)^2} \leq 2-\frac{1}{n+1}$

$$\frac{1}{1^2}+\cdots+\frac{1}{n^2}+\frac{1}{(n+1)^2} \leq 2-\frac{1}{n}+\frac{1}{(n+1)^2}$$

$$\leq 2-\frac{1}{n}+\frac{1}{n^2+2n+1}$$

$$\leq 2-\frac{(n+1)^2}{n(n+1)^2}+\frac{n}{(n+1)(n+1)n}$$ valid because $n \geq 1$

$$\leq 2-\frac{(n+1)^2+n}{n(n+1)^2}$$

$$\text{if } \cancel{2}-\frac{1}{n+1} \geq \cancel{2}-\frac{(n+1)^2+n}{n(n+1)^2}$$

$$\frac{1}{n+1} \leq \frac{(n+1)^2+n}{n(n+1)^2}$$

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