# Math 61 Midterm 2

Frank Siyuan Xing

TOTAL POINTS

**91 / 100**

QUESTION 1

1 **8 / 10**

- ✓ + **2 pts** a) False
- ✓ + **2 pts** b) False
- ✓ + **2 pts** c) False
- ✓ + **2 pts** d) True
- \+ **2 pts** e) True

QUESTION 2

30 pts

2.1 a **6 / 6**

- ✓ - **0 pts** Correct

2.2 b **4 / 4**

- ✓ - **0 pts** Correct (based on answer to a)

2.3 c **10 / 10**

- ✓ - **0 pts** Correct

2.4 d **10 / 10**

- ✓ + **10 pts** Correct
- \+ **2 pts** correct constant on (-1)^n
- \+ **2 pts** correct constant on 5^n
- \+ **6 pts** g(n)=-2
- \+ **2 pts** g(n)=c
- \- **2 pts** Minor computational error
- \+ **0 pts** No significant progress

QUESTION 3

3 **19 / 20**

- ✓ - **1 pts** Gave the length of the shortest path from a to d as the final answer

QUESTION 4

25 pts

4.1 a **10 / 10**

- ✓ + **10 pts** Correct
- \+ **0 pts** Did not draw a Hamiltonian cycle

4.2 b **15 / 15**

- ✓ + **15 pts** Correct
- \+ **0 pts** Solution lacked any significant content/tried to use counting method to conclude that no path exists (and believed it worked/used it improperly)
- \- **8 pts** Solution is vaguely worded, incomplete or includes false claims that doesn't affect final argument.
- \- **3 pts** Does not explain why certain edges must be used/excluded in a potential Hamiltonian cycle.

QUESTION 5

15 pts

5.1 a **4 / 10**

- ✓ - **6 pts** Bin 3: Incorrect argument (with false statements) but setup is decent and perhaps, with significant effort, could have been formalized into a full proof. (e.g. looked at pairs of vertices that do not have paths, and tried to contradict; or broke into two sets of 5 vertices)

5.2 b **5 / 5**

- ✓ - **0 pts** Correct

# Midterm 2 (Math 61)

## Friday, May 17th

Name: Frank Xing

ID: 905-164-685

Circle your discussion section:

| Tuesday | Thursday | |
|---|---|---|
| 2A | 2B | TA: Harris Khan |
| 2C | 2D | TA: Fred Vu |
| (2E) | 2F | TA: Matthew Stone |

This exam has 8 pages, including the cover page. Please make sure your exam includes each page. Please write your name on *each* page you submit. You will have 50 minutes to complete this exam. **You may not use a calculator**, or consult your textbook, class notes, or any other materials. If you need scratch paper or more space for your answers, please use the back of the pages.

**If there is any work on the backs of the pages which you would like to have graded, please indicate this clearly on the front of the page for the corresponding problem.**

Show your work for these problems, don't just give an answer. If a question asks you to prove something, please write a complete proof. Unless otherwise stated, you may use any results proved in class or in the textbook, but please make it clear when you are doing so. Unless otherwise stated, you will *not* receive full credit for giving the correct answer with no explanation. You may still earn partial credit even if your final answer is incorrect.

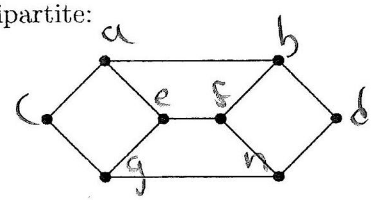
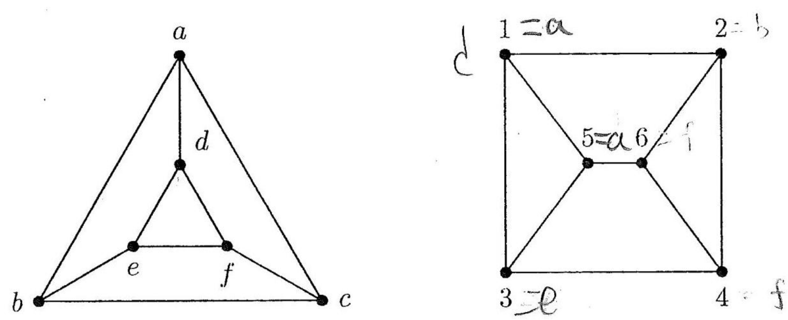
**Remember that you are bound by a conduct code, and that you may not look at anyone's paper or let anyone look at your paper.**

| Question | Points | Score |
|---|---|---|
| 1 | 10 | |
| 2 | 30 | |
| 3 | 20 | |
| 4 | 25 | |
| 5 | 15 | |
| Total: | 100 | |

Name: Frank Xing

**1.** [10 pts, 2 points each] Mark each of the following statements as either TRUE or FALSE. For this question you do not need to show any work beyond the final answer.

*Be sure to read the questions carefully!*

| | |
|---|---|
| **(a)** The complete bipartite graph $K_{61,2019}$ has an Euler cycle. | False |
| **(b)** There is a graph with 9 vertices in which every vertex has degree 3. | False |
| **(c)** The graph $G$ with adjacency matrix $A = \begin{pmatrix} 2 & 0 & 1 & 0 & 3 \\ 0 & 4 & 0 & 3 & 0 \\ 1 & 0 & 6 & 0 & 5 \\ 0 & 3 & 0 & 2 & 0 \\ 3 & 0 & 5 & 0 & 8 \end{pmatrix}$ is connected. ($v_1$) | False |
| **(d)** The graph below is bipartite:   a b / f c / d e / g h | True |
| **(e)** The two graphs below are isomorphic:   | False |

Name: Frank Xing

2. [30 pts] Solve the following recursion relations. In each case, your answer should be a formula for the $n^{th}$ term of the sequence in terms of $n$.

*Show your work. You may use results proved in class or in the textbook, but make it clear how you are getting your answers. A correct answer on its own will not be sufficient for full credit.*

(a) [6 pts] Find a *general* solution to the recursion relation $a_n = 4a_{n-1} + 5a_{n-2}$. (That is, find a formula for $a_n$ in terms of $n$ and some unknown constants, that will work for any choice of initial conditions.)

We have polynomial:

$$\lambda^2 - 4\lambda - 5 = 0$$

$$(\lambda - 5)(\lambda + 1) \rightarrow \lambda = -1 \text{ or } 5$$

So general solution:

$$a_n = C_1(-1)^n + C_2(5)^n.$$

(b) [4 pts] Find the solution to the recurrence relation $a_n = 4a_{n-1} + 5a_{n-2}$ with initial conditions $a_0 = 5$ and $a_1 = 7$.

Plugging in:

$$a_0 = C_1 + C_2 = 5$$

$$a_1 = -C_1 + 5C_2 = 7$$

$$6C_2 = 12$$

$$C_2 = 2 \qquad C_1 = 3$$

So the solution: $a_n = 3(-1)^n + 2(5)^n$.

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(c) [10 pts] Find the solution to the recurrence relation $b_n = b_{n-1}^4 b_{n-2}^5$ with initial conditions $b_0 = 1$ and $b_1 = 16$.

$b_0 = 16^0 \quad b_1 = 16^1$

We see that $b_2 = 16^4$, $b_3 = 16^{21}$, $b_4 = 16^{104}$

We see the pattern

$b_n = 16^{c_n}$ where

$c_n$ is the relation $c_0 = 0 \quad c_1 = 1$

$c_n = 5c_{n-2} + 4c_{n-1}$

$\lambda^2 - 4\lambda - 5 = 0 \rightarrow (\lambda-5)(\lambda+1) \rightarrow \lambda = -1$ ~~and~~ or $5$.

$C_1 + C_2 = 0$

$-C_1 + 5C_2 = 1$

$6C_2 = 1$ → $C_2 = \frac{1}{6}$

$C_1 = -\frac{1}{6}$

$c_n = C_1(-1)^n + C_2(5)^n = -\frac{1}{6}(-1)^n + \frac{1}{6}(5)^n$

$b_n = (16)^{\left(-\frac{1}{6}(-1)^n + \frac{1}{6}(5)^n\right)}$

(d) [10 pts] Find the solution to the recurrence relation $c_n = 4c_{n-1} + 5c_{n-2} + 16$ with initial conditions $c_0 = 0$ and $c_1 = 2$.

We solve $c_n = 4c_{n-1} + 5c_{n-2}$

Equation: $\lambda^2 - 4\lambda - 5 = 0$

$(\lambda-5)(\lambda+1) \rightarrow \lambda = -1$ ~~and~~ or $5$.

We have $c_1(-1)^n + c_2(5)^n$

then we guess the form

We guess the form $a$..

$a = 4a + 5a + 16$

$a = 9a + 16$

$-8a = 16 \Rightarrow a = -2$

We have $c_n = c_1(-1)^n + c_2(5)^n - 2$

Plug in:

$c_1 + c_2 = 2$

$-c_1 + 5c_2 = 4$

$6c_2 = 6$

$c_2 = 1$

$c_1 = 1$

$c_n = (-1)^n + (5)^n - 2$.

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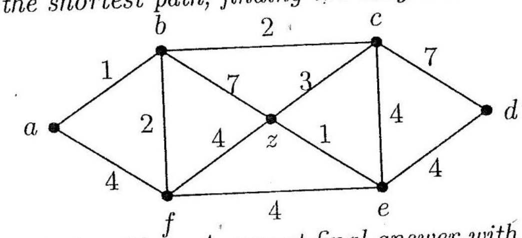
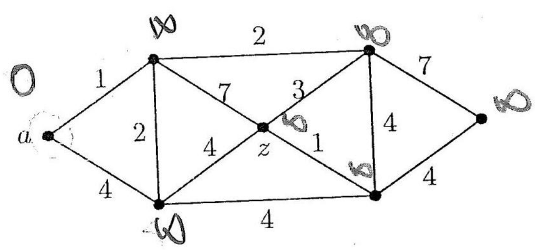
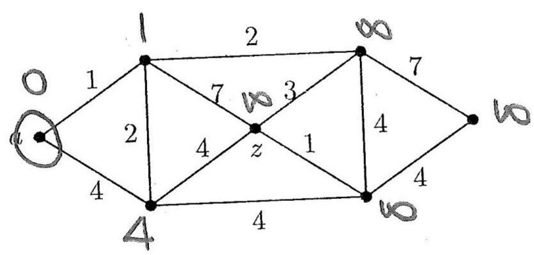
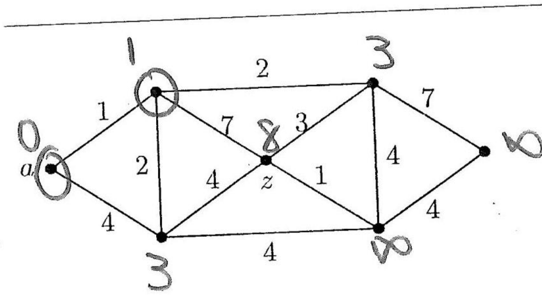
**3.** [20 pts] Use Dijkstra's algorithm to find the length of the shortest path (i.e. the path for which the sum of the labels is as small as possible) between $a$ and $z$ in the weighted graph below. *You do not need to find the shortest path, finding it's length will be sufficient.*

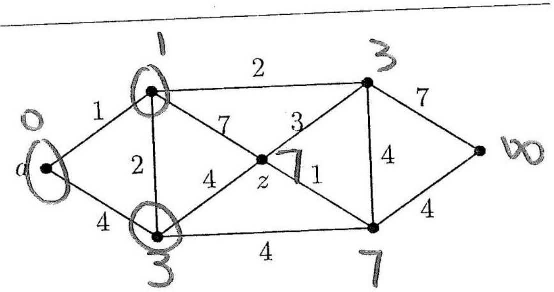


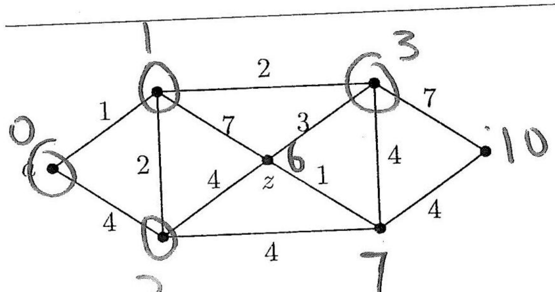

*Show each step of Dijkstra's algorithm. A correct final answer with no work shown will not be sufficient for full credit.* Use the blank graphs below for your answer. If you make a mistake, clearly cross it out and continue using the next blank graph. There are additional blank graphs on the back of this page.

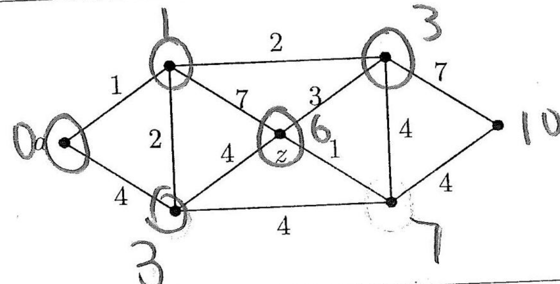






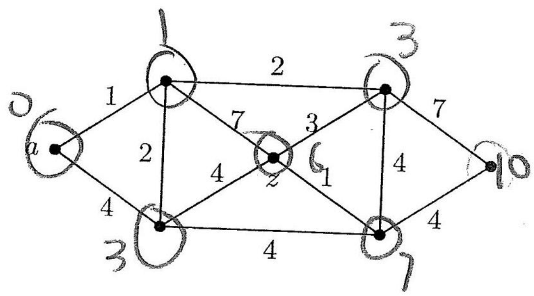






**Answer:** 10

Check this box if you used any graphs from the back of the page: ☑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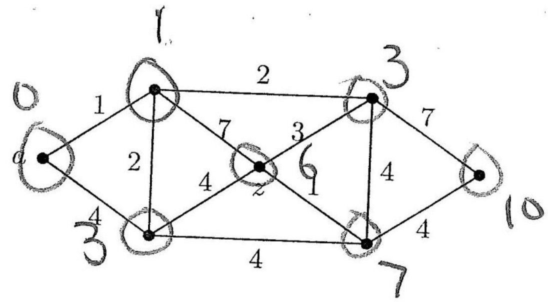
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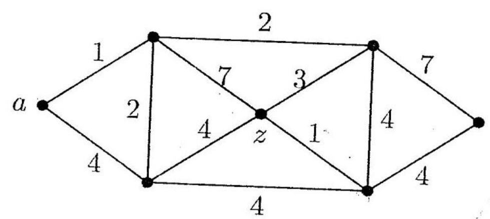
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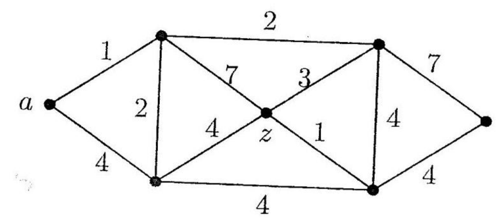
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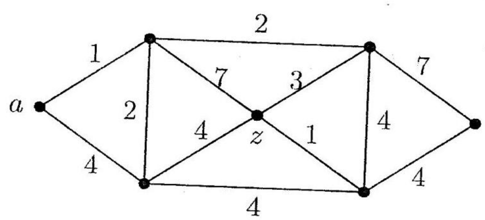
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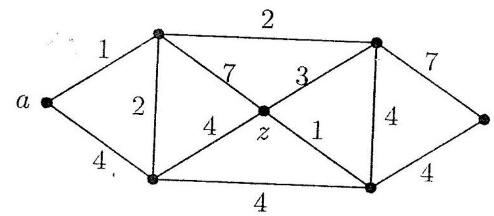
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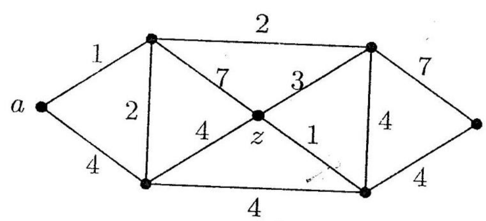
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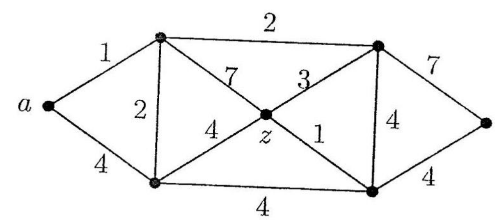
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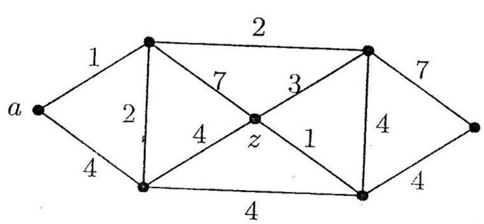
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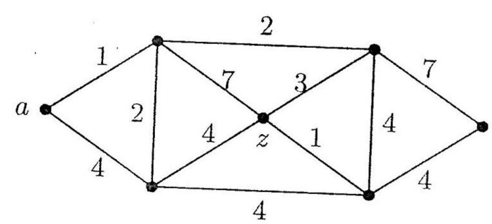
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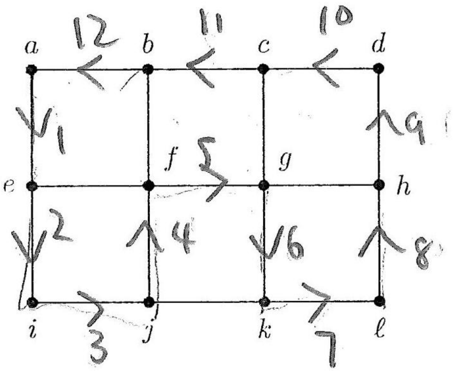
**4.** [25 pts] In each of the following graphs, either find a Hamiltonian cycle (i.e. a cycle which uses every vertex exactly once), or prove that the graph does not have a Hamiltonian cycle.

If the graph does have a Hamiltonian cycle, **CLEARLY** drawing this cycle on the provided graph (so that there's no ambiguity as to which edges are used, and in which order), or listing out the vertices in the order traveled (i.e. writing something like $(a, b, e, d, c, a)$) will be sufficient for full credit.

If the graph does not have a Hamiltonian cycle, you must give an explanation as to why. *Simply drawing diagrams with no explanation will NOT be sufficient for full credit.*

(a) [10 pts]




This graph does have a Hamiltonian cycle with the path

(a, e, i, j, f, g, k, l, h, d, c, b, a).

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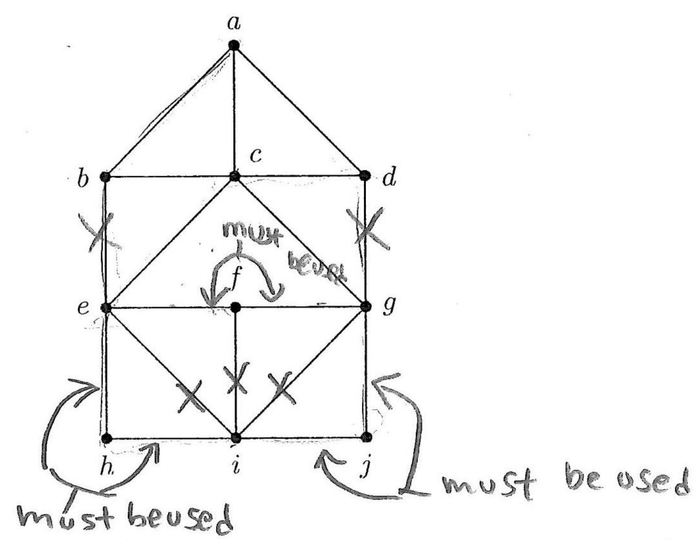
**(b)** [15 pts]




This graph doesn't have a hamiltonian cycle because we see the edges from e to i, f to i, and g to i won't be used. This means the edge e to f and f to g has to be used. Combined that with the edge from e to h and g to j that has to be used. We can no longer return from the vertex a, b, c, or d (without going through the same vertex twice), meaning there is no cycle.

Name:________________________

**5.** [15 pts]

(a) [10 pts] Let $G$ be a *simple* graph with 10 vertices, in which every vertex has degree at least 5. Prove that $G$ is connected. [Hint: You may want to use the pigeonhole principle.]

Since every vertex has degree at least 5, we will have at least $\frac{5 \cdot 10}{2}$ 25 edges.

We label the vertex $\{a_1, a_2 \cdots a_{10}\}$

We can split the vertex $\{a_1, a_2\}$ $\{a_3, a_4\}$ $\{a_5, a_6\}$, $\{a_7, a_8\}$ $\{a_9, a_{10}\}$.

Since each vertex is connected to five other vertices, $a_1$ must be connected to $a_2$ if it connects to a vertex in each other set. $a_2$ also must connect to $a_1$ if it connects with a different vertex in every other set.

The same argument could be applied to all the vertices and the graph is connected.

(b) [5 pts] Draw a *simple* graph with 10 vertices which is *not* connected, in which every vertex has degree 4.