19F-MATH61-2 Midterm 2

TOTAL POINTS

100 / 100

QUESTION 1

10 pts

1.1 a **2 / 2**

✓ **- 0 pts** Correct: True

1.2 b **2 / 2**

✓ **- 0 pts** Correct: False

1.3 c **2 / 2**

✓ **- 0 pts** Correct: False

1.4 d **2 / 2**

✓ **- 0 pts** Correct: False

1.5 e **2 / 2**

✓ **- 0 pts** Correct: True

QUESTION 2

30 pts

2.1 a **9 / 9**

✓ **- 0 pts** Correct

2.2 b **9 / 9**

✓ **- 0 pts** Correct

2.3 c **12 / 12**

✓ **- 0 pts** Correct

QUESTION 3

3 **20 / 20**

✓ **- 0 pts** Correct

QUESTION 4

4 **20 / 20**

✓ **- 0 pts** Correct

QUESTION 5

20 pts

5.1 a **4 / 4**

✓ **- 0 pts** Correct

5.2 b **6 / 6**

✓ **+ 2 pts** S_n Correct

✓ **+ 4 pts** T_n Correct

+ **0 pts** S_n Incorrect/No progress

+ **0 pts** T_n Incorrect/No Progress

+ **0 pts** Shifted indices to get T_n, instead of correct computation

+ **0 pts** Blank

+ **0 pts** Didn't do T_n

+ **1 pts** Started S_n, but couldn't finish

+ **1 pts** Started T_n, but couldn't finish

+ **3 pts** Need more justification on computations for T_n

+ **2 pts** Algebra error on T_n

+ **3 pts** Minor algebra error on T_n

5.3 c **10 / 10**

✓ **- 0 pts** Correct

Midterm 2 (Math 61)

Monday, November 18th

This exam has 6 (double sided) pages, including the cover page, and some blank pages at the end. Please make sure your exam includes each page. Please write your name on *each* page you submit. You will have 50 minutes to complete this exam. **You may not use a calculator**, or consult your textbook, class notes, or any other materials. If you need scratch paper or more space for your answers, please use the extra page at the end.

If there is any work on the blank pages which you would like to have graded, please indicate this CLEARLY on the page for the corresponding problem.

Show your work for these problems, don't just give an answer. If a question asks you to prove something, please write a complete proof. Unless otherwise stated, you may use any results proved in class or in the textbook, but please make it clear when you are doing so. Unless otherwise stated, you will *not* receive full credit for giving the correct answer with no explanation. You may still earn partial credit even if your final answer is incorrect.

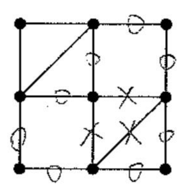
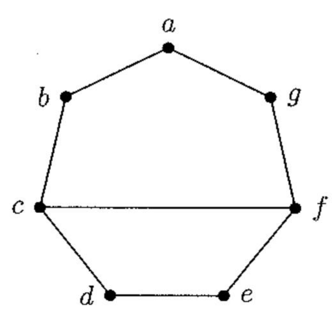
Remember that you are bound by a conduct code, and that you may not look at anyone's paper or let anyone look at your paper.

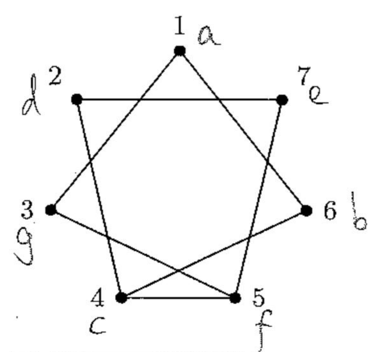
Question	Points	Score
1	10	
2	30	
3	20	
4	20	
5	20	
Total:	100	

1. [10 pts, 2 points each] Mark each of the following statements as either TRUE or FALSE. For this question you do not need to show any work beyond the final answer.

Be sure to read the questions carefully! $t=-3,\ t=2$ $\quad C_1 2^n + \cancel{C_2(-3)^n}$

$t^2+t-6=0 \quad (t+3)(t-2)=0$

<table>
<tr><td>(a) The recurrence relation $a_n = -a_{n-1} + 6a_{n-2}$ has a solution which is strictly increasing (i.e. satisfies $a_{n+1} > a_n$ for all $n \geq 0$).</td><td>T</td></tr>
<tr><td>(b) A graph with 8 vertices must have at least 4 edges.</td><td>F</td></tr>
<tr><td>(c) If a graph has a Hamiltonian cycle, then it must also have an Euler cycle.</td><td>F</td></tr>
<tr><td>(d) The graph below has a Hamiltonian cycle:
</td><td>F</td></tr>
<tr><td>(e) The two graphs below are isomorphic:



</td><td>T</td></tr>
</table>

2. [30 pts] In each of the following problems, either draw a graph with the listed properties, or prove that no such graph can exist.

In parts (a) and (b), if such a graph does exist, **CLEARLY** drawing this graph will be sufficient for full credit (in part (c), you will need to give some extra information in your picture).

If such a graph does not exist, you must give a full explanation as to why. *Simply drawing diagrams with no explanation will NOT be sufficient for full credit.*

5 6 9

(a) [9 pts] A simple graph with 6 vertices of degrees 1, 2, 2, 3, 3 and 4.

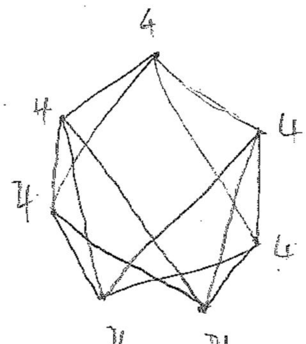
$1+2+2+3+3+4=15$

15 is odd.

Sum of degrees must be even

so not possible.

(b) [9 pts] A simple graph with 7 vertices, all of which have degree 4.



so it exists

(c) [12 pts] A simple *bipartite* graph with 7 vertices which has a Hamiltonian cycle.

If such a graph does exist, you must **CLEARLY** draw it *and* indicate what the Hamiltonian cycle is and how to partition the vertices to demonstrate that the graph is bipartite. Doing this will be sufficient for full credit.

If such a graph does not exist, you must give a full explanation as to why. *Simply drawing diagrams with no explanation will NOT be sufficient for full credit.*

Not possible.

If we partition the graph into set V_1 and V_2,

each edge is incident on one $V_i \in V_1$ and another $V_j \in V_2$.

If it has a hamiltonian cycle, it will have 7 vertices and 7 edges.

So it will be from $V_1 \xrightarrow[1]{} V_2 \xrightarrow[2]{} V_1 \xrightarrow[3]{} V_2 \xrightarrow[4]{} V_1 \xrightarrow[5]{} V_2 \xrightarrow[6]{} V_1 \xrightarrow[7]{} V_2$

or $V_2 \xrightarrow[1]{} V_1 \xrightarrow[2]{} V_2 \xrightarrow[3]{} V_1 \xrightarrow[4]{} V_2 \xrightarrow[5]{} V_1 \xrightarrow[6]{} V_2 \xrightarrow[7]{} V_1$

It will always land in the other partition of vertices with 7 edges

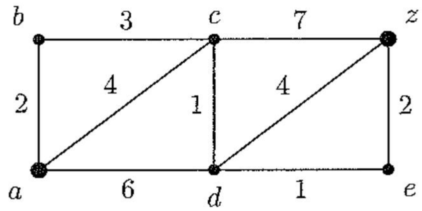
So no hamiltonian cycle exists

So, with only 7 edges, it can never go back on a cycle so it eventually lands in another partition and $V_1 \cap V_2 = \emptyset$

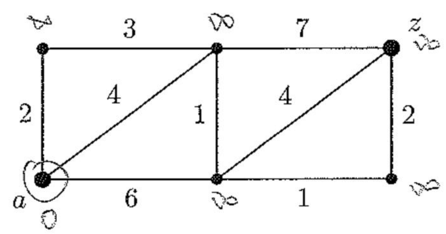
(if we start with $V_i \in V_1$, then any $V_j \in V_2 \neq V_i$;

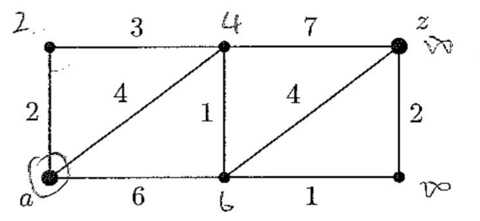
if we start with $V_j \in V_2$, then any $V_i \in V_1 \neq V_j$)

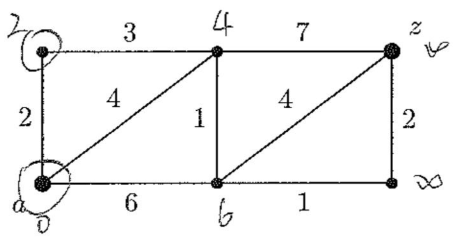
3. [20 pts] Use Dijkstra's algorithm to find the length of the shortest path (i.e. the path for which the sum of the labels is as small as possible) between a and z in the weighted graph below. *You do not need to find the shortest path, finding it's length will be sufficient.*



Show each step of Dijkstra's algorithm on a separate graph. It should be clear what all of the labels were at each step of the algorithm, and which vertex was circled in each step. A correct final answer with no work shown will not be sufficient for full credit. Use the blank graphs below for your answer. If you make a mistake, clearly cross it out and continue using the next blank graph. There are additional blank graphs on the back of this page.

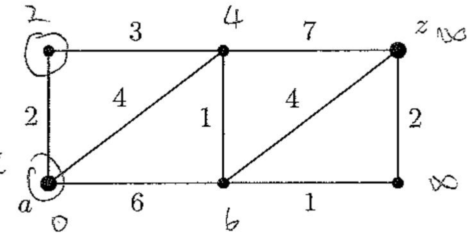


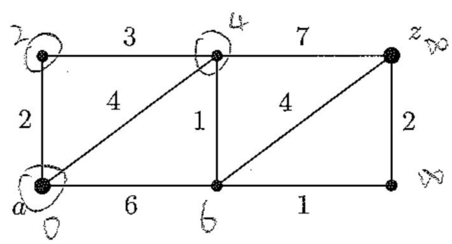


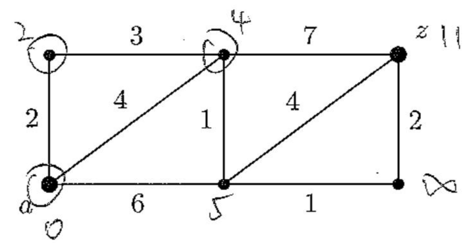


nothing

to update



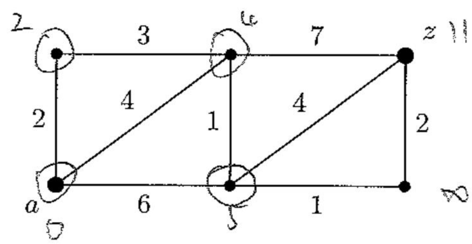




Answer: 8

Check this box if you used any graphs from the back of the page: ✓

Name:______________________________



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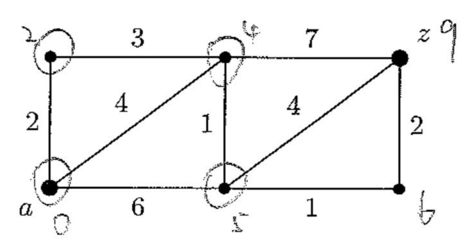

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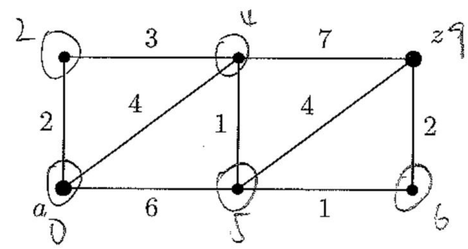

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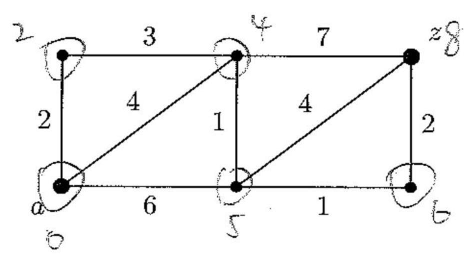

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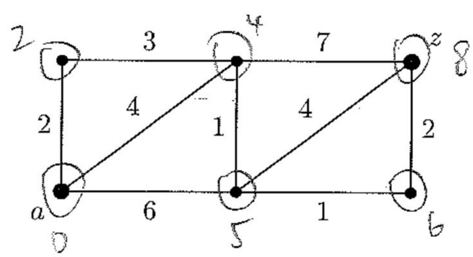

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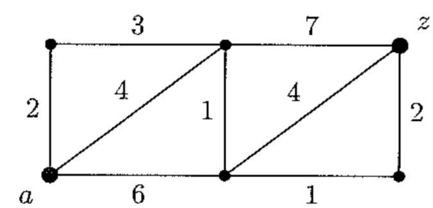

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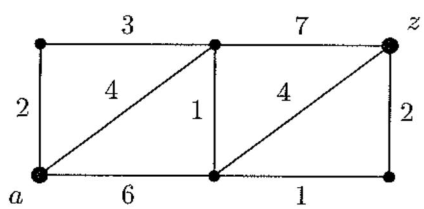

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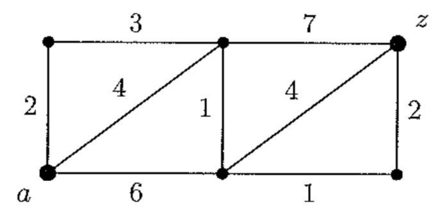

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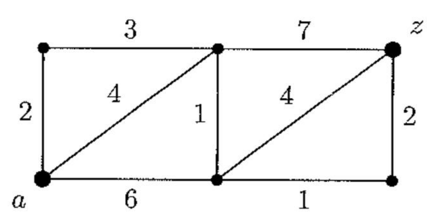

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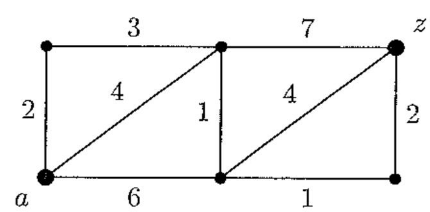

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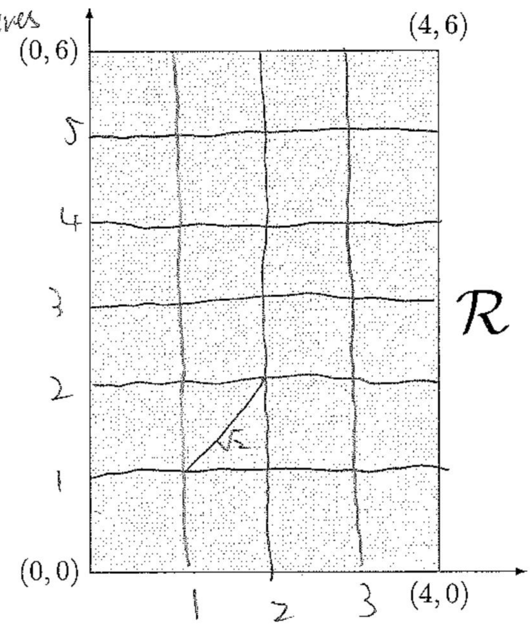
4. [20 pts] 25 points $P_1, P_2, \ldots, P_{25}$ are selected within the rectangle $\mathcal{R}$ with vertices at $(0,0), (4,0), (4,6)$ and $(0,6)$. Prove that some two of these points have a distance of $\sqrt{2}$ or less (that is, prove that there are some i and j with $i \neq j$ and the distance from P_i to P_j is $\sqrt{2}$ or less). [Hint: Try subdividing $\mathcal{R}$ into 24 different regions and using the pigeonhole principle.]

Recall the *distance formula*: If $P_i = (x_i, y_i)$ and $P_j = (x_j, y_j)$ then the distance from P_i to P_j is $\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$.

I will partition the rectangle into squares of unit length 1



Maximum distance within each square is $\sqrt{2}$, which is the diagonal of the 1x1 square with length $\sqrt{1^2+1^2}$.

We get 24 squares from this partition, and we have 24 squares,

At least 2 of them falls into the same square (including the sides and vertices of the squares) according to pigeonhole principle.

Worst case scenario: Those 2 points falls onto the diagonal vertices of the small square, which they will have distance $\sqrt{2}$.
else, their distance $< \sqrt{2}$.

So the 2 points that fall into the same square has distance $\leq \sqrt{2}$.

5. [20 pts] In this problem, we will consider sequences satisfying the following homogeneous linear recursion relation with *non-constant* coefficients:

$$a_n = 2na_{n-1} - n(n-1)a_{n-2}, \qquad \text{for } n \geq 2 \qquad (\star)$$

In this problem, you may use the following fact without proof:

If A_n and B_n are two solutions to ($\star$) (i.e. $A_n = 2nA_{n-1} - n(n-1)A_{n-2}$ and $B_n = 2nB_{n-1} - n(n-1)B_{n-2}$) satisfying $A_0 = B_0$ and $A_1 = B_1$, then $A_n = B_n$ for all $n \geq 0$.

(a) [4 pts] If A_n and B_n are two solutions to ($\star$) and x and y are two real numbers, then prove that the sequence $C_n = xA_n + yB_n$ is also a solution to ($\star$) (i.e. prove that $C_n = 2nC_{n-1} - n(n-1)C_{n-2}$ for all $n \geq 2$).

we need to prove $xA_n + yB_n = 2nC_{n-1} - n(n-1)C_{n-2}$

proof: $LHS = x(2nA_{n-1} - n(n-1)A_{n-2}) + y(2nB_{n-1} - n(n-1)B_{n-2})$

$= x\cdot 2nA_{n-1} + y\cdot 2nB_{n-1} - xn(n-1)A_{n-2} - yn(n-1)B_{n-2}$

$RHS = 2n(xA_{n-1} + yB_{n-1}) - n(n-1)(xA_{n-2} + yB_{n-2})$

$= x\cdot 2nA_{n-1} + y\cdot 2nB_{n-1} - xn(n-1)A_{n-2} - yn(n-1)B_{n-2}$ //

$LHS = RHS$. $\leftarrow C_0 = xA_0 + yB_0$

Q.E.D)

(b) [6 pts] Prove that the sequences $S_n = n!$ and $T_n = (n+1)!$ are both solutions to ($\star$) (i.e. prove that $S_n = 2nS_{n-1} - n(n-1)S_{n-2}$ and $T_n = 2nT_{n-1} - n(n-1)T_{n-2}$ for all $n \geq 2$).

① to prove $S_n = 2nS_{n-1} - n(n-1)S_{n-2}$

$LHS = n!$

$RHS = 2n(n-1)! - n(n-1)(n-2)!$

$= 2n! - n! = n!$

$LHS = RHS$

② to prove $T_n = 2nT_{n-1} - n(n-1)T_{n-2}$

$LHS = (n+1)!$

$RHS = 2n\,n! - n(n-1)(n-1)!$

$= 2n\,n! - (n-1)\,n!$

$= (2n - n + 1)\,n!$

$= (n+1)\,n!$ $\quad LHS = RHS$ $\quad$ Q.E.D

$= (n+1)!$

(c) [10 pts] The sequence b_n is defined by the initial conditions $b_0 = 5$ and $b_1 = 7$, and the recursion relation ($\star$), i.e

$$b_n = 2nb_{n-1} - n(n-1)b_{n-2}$$

for $n \geq 2$. Find a formula for b_n as a function of n (i.e. solve this recursion), *and prove that your answer is correct.* [Hint: Think about how we came up with our general solution to a linear homogeneous recurrence relation with constant coefficients. The recursion doesn't have constant coefficients, but you may be able to use a similar strategy.]

We now have 2 linearly independent solutions $S_n = n!$, $T_n = (n+1)!$

suppose $b_n = C_1 n! + C_2 (n+1)!$

when $n=0$, $b_0 = 5$

$5 = C_1 + C_2$

when $n=1$, $b_1 = 7$

$7 = C_1 + 2C_2$

$C_2 = 2$, $C_1 = 3$

$b_n = 3n! + 2(n+1)!$

To prove that this is correct:

$RHS = 2n b_{n-1} - n(n-1) b_{n-2}$

$= 2n\left[3(n-1)! + 2n!\right] - n(n-1)\left[3(n-2)! + 2(n-1)!\right]$

$= 6n! + 4nn! - 3n! - 2(n-1)n!$

$= 3n! + (4n - 2n + 2)n!$

$= 3n! + (2n+2)n!$

$= 3n! + 2(n+1)n!$

$= 3n! + 2(n+1)!$

$= LHS$.

Q.E.D.

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