

Math 61 Midterm 1

Derek Hu

TOTAL POINTS

89 / 100

QUESTION 1

1 8 / 10

- + 2 pts a: True
- ✓ + 2 pts b: False
- ✓ + 2 pts c: True
- ✓ + 2 pts d: True
- ✓ + 2 pts e: False

QUESTION 2

35 pts

2.1 a 10 / 10

- ✓ - 0 pts Correct 3!6!

2.2 b 10 / 10

- ✓ - 0 pts Correct $3^6 \cdot 2^4 \cdot 10! / (4!6!)$

2.3 c 6 / 15

- ✓ - 1 pts Didn't simplify combinations in final answer
- ✓ - 8 pts Incorrect/unsuccessful use of PIE/counting complements etc. to count the number of ways to place the 5s

QUESTION 3

3 15 / 15

- ✓ + 5 pts Showed reflexivity
- ✓ + 5 pts Showed symmetry
- ✓ + 5 pts Showed transitivity
- + 0 pts No credit due

QUESTION 4

4 20 / 20

- ✓ - 0 pts Correct

QUESTION 5

20 pts

5.1 a 10 / 10

- + 0 pts No serious progress / didn't understand the relevant definitions
- + 3 pts Tried something somewhat sensible, but did not make any serious progress towards a formal proof.
- + 6 pts Started with $f(x) = f(y)$, applied g to both sides, concluded $h(x) = h(y)$, but could not properly conclude the result (or struggled to articulate it formally).

✓ + 10 pts Correct. Started with $f(x) = f(y)$, applied g to both sides, concluded $h(x) = h(y)$, and used injectivity of h to conclude $x = y$. (Or, ran an equivalent proof by contradiction).

5.2 b 10 / 10

- + 0 pts No serious progress
- + 2 pts Bin 4: finite X, Y, Z , one of them is not a function, or g is injective or h is not injective or the composition is not done correctly. Or definitions missing
- ✓ + 10 pts Bin 5: finite X, Y, Z with f, g, h written out as sets or picture, g clearly not injective and h clearly injective
- + 2 pts Bin 6: some complicated f, g not satisfying the basic conditions necessary for the counterexample (or not at all justified to have these properties)
- + 5 pts Bin 1: g is simple, f is complicated. things like x^2 for g , complicated function for f , X, Y, Z not specified, and h not proven to be injective and nontrivial to see that it is
- + 8 pts Bin 2: both f, g are simple. $x^2, x^{1/2}$, and worked out why h is injective (or it is clear) and why g is not. Left out what X, Y, Z should be (or incorrectly

specified them)

+ **10 pts** Bin 3: Fully correct example (X,Y,Z specified or drawn, g not injective and h injective)

MIDTERM 1 (MATH 61)

MONDAY, APRIL 22ND

Name: Derek Kim

ID: 805 110 848

Circle your discussion section:

Tuesday Thursday

2A **2B** TA: Harris Khan

2C 2D TA: Fred Vu

2E 2F TA: Matthew Stone

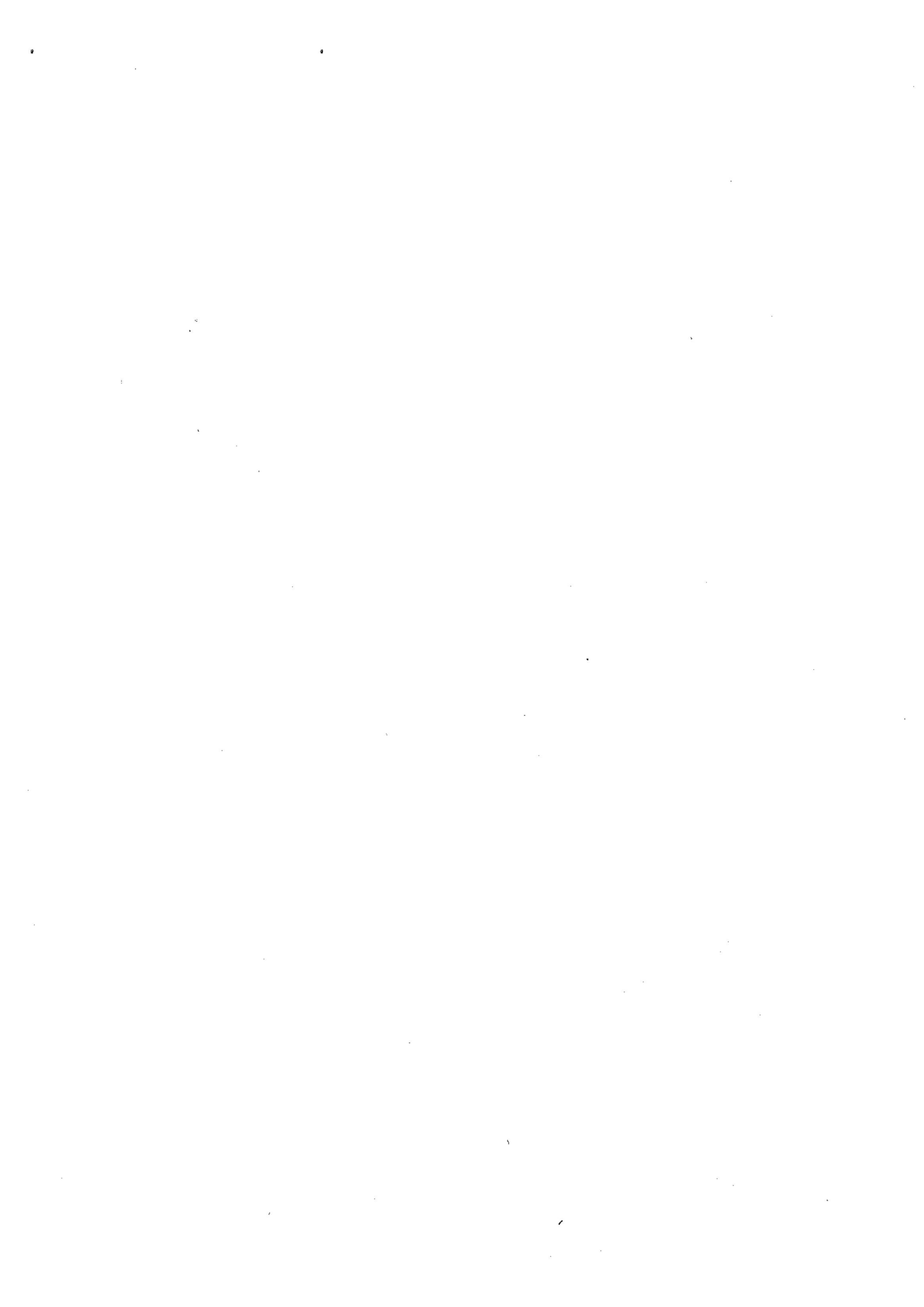
This exam has 7 pages, including the cover page. Please make sure your exam includes each page. Please write your name on *each* page you submit. You will have 50 minutes to complete this exam. **You may not use a calculator**, or consult your textbook, class notes, or any other materials. If you need scratch paper or more space for your answers, please use the back of the pages.

If there is any work on the backs of the pages which you would like to have graded, please indicate this clearly on the front of the page for the corresponding problem.

Show your work for these problems, don't just give an answer. If a question asks you to prove something, please write a complete proof. Unless otherwise stated, you may use any results proved in class or in the textbook, but please make it clear when you are doing so. Unless otherwise stated, you will *not* receive full credit for giving the correct answer with no explanation. You may still earn partial credit even if your final answer is incorrect.

Remember that you are bound by a conduct code, and that you may not look at anyone's paper or let anyone look at your paper.

Question	Points	Score
1	10	
2	35	
3	15	
4	20	
5	20	
Total:	100	



Name: _____

1. [10 pts, 2 points each] Mark each of the following statements as either TRUE or FALSE. For this question you do not need to show any work beyond the final answer.

Be sure to read the questions carefully!

(a) The set $\{\{1\}, 3, \{2, 3\}, 2, 3\}$ has cardinality 4.	False
(b) Every relation is either symmetric or antisymmetric.	False
(c) If R is a relation satisfying $R = R^{-1}$, then R is symmetric.	True
(d) The set $f = \{(C, \clubsuit), (B, \heartsuit), (E, \heartsuit), (D, \spadesuit), (A, \clubsuit)\}$ is a function from $X = \{A, B, C, D, E\}$ to $Y = \{\diamond, \clubsuit, \heartsuit, \spadesuit\}$.	True
(e) The set $X = \{1, 2, 3, 4, 5, 6\}$ has more subsets of cardinality 4 than subsets of cardinality 3.	False

b) $\{(1, 2), (2, 3), (3, 2)\}$

c) If $(x, y) \in R$, then $(y, x) \in R^{-1}$. Since $R = R^{-1}$, $(y, x) \in R$ is also true.

e) ${}^6C_4 = \frac{6!}{2!4!} = \frac{6 \cdot 5}{2} = 15$

${}^6C_3 = \frac{6!}{3!3!} = \frac{6 \cdot 5 \cdot 4}{6} = 20$

$$\begin{array}{cccc}
 & & 1 & \\
 & & 2 & 1 \\
 & 1 & 3 & 3 & 1 \\
 & 1 & 4 & 6 & 4 & 1 \\
 \binom{4}{4} & \binom{4}{3} & \binom{4}{2} & \binom{4}{1} & \binom{4}{0}
 \end{array}$$

$$\frac{4!}{2!2!}$$

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2. [35 pts] Compute the following quantities. You may leave your answers in terms of exponents and factorials, but do not leave your final answers in terms of $P(n, r)$ or $C(n, r)$ (so $4^{12} \frac{15!}{3!6!2!}$ would be an acceptable final answer, but $P(10, 3)C(18, 7)$ would not).

Show your work. It should be clear how you got your answers.

- (a) [10 pts] The number of permutations of the letters *COMPUTER* that contain the letters *CPU* together in any order (so for instance, *MTUPCREO* would be one such arrangement, but *PMTCOREU* would not).

Let the letters "CPU" be a single unit. Now
 Here are 6 units to order. (CPU) O M T E R

— — — — — There are $6!$ to order
 these 6 units and within each ordering there are

$3!$ to order the unit containing CPU (e.g. CUP, PCU, etc.)

$$\boxed{6! \cdot 3!}$$

- (b) [10 pts] The coefficient of $x^6 y^8$ in the expansion of $(3x + 2y^2)^{10}$. [Don't forget that y is squared in this expression.]

$$x^6 (y^2)^4 \quad \text{Let } a = 3x \quad b = 2y^2$$

We want to find the coefficient for the term $a^6 b^4$

By the binomial theorem, the coefficient for $a^6 b^4$ is
 $C(10, 6) = \frac{10!}{4!6!}$ a and b have coefficients themselves

$$a^6 = 3^6 x^6 \quad b^4 = 2^4 y^8$$

So the entire term is $\frac{10!}{4!6!} 3^6 \cdot 2^4 x^6 y^8$

$$\boxed{\text{Coefficient: } \frac{10!}{4!6!} 3^6 \cdot 2^4}$$

Name: _____

- (c) [15 pts] A standard 6-sided die (with sides numbered 1-6) is rolled 10 times in a row. How many possible outcomes are there in which exactly four 5's were rolled, and no two 5's were ever rolled in a row.

[So for example 4562251565 would be one such outcome, but 2544551533 would not. Also order matters, so 4562251565 would be considered a different outcome from 5254152566.]

Create a slot

1	2	3	4	5	6	7	8	9	10
		5		5		5			

$C(10,4)$ ways to choose 4 slots for

the

(Subtract the ones w/ 2 fives in a row $9 \cdot C(8,2)$)

The other 6 slots each have 5 possibilities,

$$(C(10,4) - 9 \cdot C(8,2)) \cdot 5^6$$

Name: _____

3. [15 pts] Let $\mathbb{R}^+ = \{x \in \mathbb{R} | x > 0\}$ be the set of positive real numbers. Define a relation R on \mathbb{R}^+ by $(x, y) \in R$ if x/y is a rational number. Prove that R is an equivalence relation.

R is an equivalence relation if it is reflexive, symmetric, and transitive.

• For all $x \in \mathbb{R}^+$, $x/x = 1$ so $(x, x) \in R$, R is reflexive. ✓

• Suppose $(x, y) \in R$. By definition this means x/y is a rational number, meaning it can be expressed as a fraction. $x/y = \frac{1}{(y/x)}$. Since 1 and (x/y) are both rational numbers, then their quotient y/x is also a rational number. Since y/x is a rational number $(y, x) \in R$, so R is symmetric.

• Suppose $(x, y), (y, z) \in R$. This means $\frac{x}{y}$ and $\frac{y}{z}$ are both rational numbers. Their product, $\frac{x}{y} \cdot \frac{y}{z} = \frac{x}{z}$, must also be a rational number. Since $\frac{x}{z}$ is a rational number, $(x, z) \in R$; so R is transitive.

Name: _____

4. [20 pts] Prove that for any positive integer n :

$$1(1!) + 2(2!) + 3(3!) + \dots + n(n!) = (n+1)! - 1$$

[Hint: Use induction.]

Base case: let $n=1$ $1(1!) = (1+1)! - 1$
 $1 = 2 - 1 \quad \checkmark$

Inductive step: Assume that the formula is true for some $n=k$.

$$1(1!) + 2(2!) + \dots + k(k!) = (k+1)! - 1$$

For $n=k+1$: $1(1!) + 2(2!) + \dots + k(k!) + (k+1)(k+1)! = (k+1)! - 1 + (k+1)(k+1)!$
 $= 1(1!) + 2(2!) + \dots + k(k!) + (k+1)(k+1)! = (k+1)! (1 + k+1) - 1$
 $= (k+2)! - 1$

Plugging in $n=k+1$ to the right side of the formula we get $(k+2)! - 1$ which matches the expression above.

Since the base case $n=1$ is true, and the formula is true for all $k+1$, the formula is true by induction.

Name: _____

5. [20 pts] Let X, Y and Z be sets, and let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be functions, and let $h = g \circ f$ be the composition of f and g . (That is, h is the function from X to Z defined by $h(x) = g(f(x))$.)

(a) [10 pts] Prove that if h is one-to-one, then f is one-to-one as well. [Hint: Assume that $f(x) = f(y)$, and try to use that to prove that $x = y$.]

Suppose $f(x) = f(y)$. This means that $g(f(x)) = g(f(y))$,
since g is a function & $f(x) = f(y)$. Since h is
defined as a function from $X \rightarrow Z$ and is one-to-one,
if $h(x) = h(y)$, then $x = y$. We know that

$h(x) = g(f(x)) = g(f(y)) = h(y)$, so $x = y$. Given that
 $f(x) = f(y)$ and that $x = y$, f is one to one

(b) [10 pts] Give a counterexample to show that it is possible for h to be one-to-one while g is not one-to-one. That is, give examples of sets X, Y and Z and functions $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ such that $h = g \circ f$ is one-to-one, but g is not one-to-one.

$$X = \{1\} \quad Y = \{2, 3\} \quad Z = \{4\}$$

$$f = \{(1, 2)\} \quad X \rightarrow Y$$

$$g = \{(2, 4), (3, 4)\} \quad Y \rightarrow Z \quad (\text{Not one-to-one})$$

$$g \circ f = \{(1, 4)\} \quad \text{one to one}$$

