Math 61 Midterm 1

Ryan Kourosh Riahi

TOTAL POINTS

85 / 100

QUESTION 1

1 8 / 10

+ **2 pts** a: True
✓ + **2 pts** b: False
✓ + **2 pts** c: True
✓ + **2 pts** d: True
✓ + **2 pts** e: False

QUESTION 2

35 pts

2.1 a 6 / 10

✓ - **4 pts** Correct, no explanation

2.2 b 10 / 10

✓ - **0 pts** Correct 3^6*2^4*10!/(4!6!)

2.3 c 6 / 15

✓ - **9 pts** Incorrect answer and unclear/incorrect reasoning for counting the number of ways to place the 5s

QUESTION 3

3 15 / 15

✓ + **5 pts** Showed reflexivity
✓ + **5 pts** Showed symmetry
✓ + **5 pts** Showed transitivity
+ **0 pts** No credit due

QUESTION 4

4 20 / 20

✓ - **0 pts** Correct

QUESTION 5

20 pts

5.1 a 10 / 10

+ **0 pts** No serious progress / didn't understand the relevant definitions

+ **3 pts** Tried something somewhat sensible, but did not make any serious progress towards a formal proof.

+ **6 pts** Started with f(x) = f(y), applied g to both sides, concluded h(x) = h(y), but could not properly conclude the result (or struggled to articulate it formally).

✓ + **10 pts** Correct. Started with f(x) = f(y), applied g to both sides, concluded h(x) = h(y), and used injectivity of h to conclude x = y. (Or, ran an equivalent proof by contradiction).

5.2 b 10 / 10

+ **0 pts** No serious progress

+ **2 pts** Bin 4: finite X,Y,Z, one of them is not a function, or g is injective or h is not injective or the composition is not done correctly. Or definitions missing

✓ + **10 pts** Bin 5: finite X,Y,Z with f,g,h written out as sets or picture, g clearly not injective and h clearly injective

+ **2 pts** Bin 6: some complicated f,g not satisfying the basic conditions necessary for the counterexample (or not at all justified to have these properties)

+ **5 pts** Bin 1: g is simple, f is complicated. things like x^2 for g, complicated function for f, X,Y,Z not specified, and h not proven to be injective and nontrivial to see that it is

+ **8 pts** Bin 2: both f,g are simple. x^2, x^1/2, and worked out why h is injective (or it is clear) and why g is not. Left out what X, Y, Z should be (or incorrectly specified them)

+ **10 pts** Bin 3: Fully correct example (X,Y,Z specified or drawn, g not injective and h injective)

Midterm 1 (Math 61)

Monday, April 22nd

Name: Ryan Riahi

ID: 105138860

Circle your discussion section:

Tuesday	Thursday	
(2A)	2B	TA: Harris Khan
2C	2D	TA: Fred Vu
2E	2F	TA: Matthew Stone

This exam has 7 pages, including the cover page. Please make sure your exam includes each page. Please write your name on *each* page you submit. You will have 50 minutes to complete this exam. **You may not use a calculator**, or consult your textbook, class notes, or any other materials. If you need scratch paper or more space for your answers, please use the back of the pages.

If there is any work on the backs of the pages which you would like to have graded, please indicate this clearly on the front of the page for the corresponding problem.

Show your work for these problems, don't just give an answer. If a question asks you to prove something, please write a complete proof. Unless otherwise stated, you may use any results proved in class or in the textbook, but please make it clear when you are doing so. Unless otherwise stated, you will *not* receive full credit for giving the correct answer with no explanation. You may still earn partial credit even if your final answer is incorrect.

Remember that you are bound by a conduct code, and that you may not look at anyone's paper or let anyone look at your paper.

Question	Points	Score
1	10	
2	35	
3	15	
4	20	
5	20	
Total:	100	

Name:________________________

1. [10 pts, 2 points each] Mark each of the following statements as either TRUE or FALSE. For this question you do not need to show any work beyond the final answer.

Be sure to read the questions carefully!

(a) The set $\{\{1\}, 3, \{2, 3\}, 2, 3\}$ has cardinality 4.	False
(b) Every relation is either symmetric or antisymmetric.	False
(c) If R is a relation satisfying $R = R^{-1}$, then R is symmetric.	True
(d) The set $f = \{(C, \clubsuit), (B, \heartsuit), (E, \heartsuit), (D, \spadesuit), (A, \clubsuit)\}$ is a function from $X = \{A, B, C, D, E\}$ to $Y = \{\diamondsuit, \clubsuit, \heartsuit, \spadesuit\}$.	True
(e) The set $X = \{1, 2, 3, 4, 5, 6\}$ has more subsets of cardinality 4 than subsets of cardinality 3.	False

Name:______________________________

2. [35 pts] Compute the following quantities. You may leave your answers in terms of exponents and factorials, but do not leave your final answers in terms of $P(n,r)$ or $C(n,r)$ (so $4^{12}\frac{15!}{3!6!2!}$ would be an acceptable final answer, but $P(10,3)C(18,7)$ would not).

Show your work. It should be clear how you got your answers.

(a) [10 pts] The number of permutations of the letters *COMPUTER* that contain the letters *CPU* together in any order (so for instance, *MTUPCREO* would be one such arrangement, but *PMTCOREU* would not).

CPU, O, M, T, E, R

$6! \cdot 3! =$

$\boxed{6! \cdot 6}$

(b) [10 pts] The coefficient of x^6y^8 in the expansion of $(3x+2y^2)^{10}$. [Don't forget that y is squared in this expression.]

$\frac{10!}{9!}$

$\frac{10!}{10!}$ (10, 1)

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

$x^2 + 2xy + y^2$

$a = 3x$

$b = 2y^2$

$(a+b)^{10}$

$a^6 b^4 = 3^6 \cdot 2^4 \cdot x^6 y^8$

Coefficient $= \dfrac{3^6 \cdot 2^4 \cdot 10!}{6! \cdot 4!}$

$C(10,4) = \dfrac{10!}{6! \cdot 4!}$

Name:________________________

(c) [15 pts] A standard 6-sided die (with sides numbered 1-6) is rolled 10 times in a row. How many possible outcomes are there in which *exactly* four 5's were rolled, and no two 5's were ever rolled in a row.

[So for example 4562251565 would be one such outcome, but 2544551533 would not. Also order matters, so 4562251565 would be considered a different outcome from 5254152566.]

5^6 total diff nums

10! Permutations

$5^6 \cdot 10! - 8! \cdot 5^6$

5 5 # # # # # # 55

8! ways to arrange $\cdot$ 5^6 permutations of nums.

Name:__________________________

3. [15 pts] Let $\mathbb{R}^+ = \{x \in \mathbb{R} | x > 0\}$ be the set of positive real numbers. Define a relation R on $\mathbb{R}^+$ by $(x, y) \in R$ if x/y is a rational number. Prove that R is an equivalence relation.

R is reflexive because $(x,x) \in R$ due to the fact that x/x is always $=1$, & 1 is a rational number

R is symmetric because if $(x,y) \in R$, then x/y is a rational number, therefore, its inverse $\left(\frac{x}{y}\right)^{-1} = \frac{y}{x}$ must also be rational, since the inverse of a rational num is a rational num, so $(y,x) \in R$

R is transitive because if $(x,y) \in R$ & $(y,z) \in R$, then $\frac{x}{y} = \lambda_1$ & $\frac{y}{z} = \lambda_2$, where λ_1 & λ_2 are rational.
so, $y = z\lambda_2$, & $\frac{x}{y} = \frac{x}{z\lambda_2} = \lambda_1$, so $\frac{x}{z} = \lambda_2 \cdot \lambda_1$, & since the product of two rationals is always rational, then $(x,z) \in R$

therefore, R is an equivalence relation

$(n+1)((n+1)!) + (n+1)! - 1$

Name:______________________________

4. [20 pts] Prove that for any positive integer n:

$$1(1!) + 2(2!) + 3(3!) + \cdots + n(n!) = (n+1)! - 1$$

[Hint: Use induction.]

Base case $n=1$

$(1)(1!) = 1$

$(1+1)! - 1 = 2 - 1 = 1$ ✓

Assume $1(1!) + 2(2!) + \cdots + n(n!) = (n+1)! - 1$ holds true for n

Prove, holds true for $n+1$

$(n+1+1)! - 1 = (n+2)! - 1 = (n+2)(n+1)! - 1$

$= ((n+1)+1)(n+1)! - 1 =$

$(n+1)((n+1)!) + (n+1)! - 1$

last element of the sequence

equal to $1(1!) + 2(2!) + \cdots + n(n!)$ by assumption

Therefore, the hypothesis is true by induction

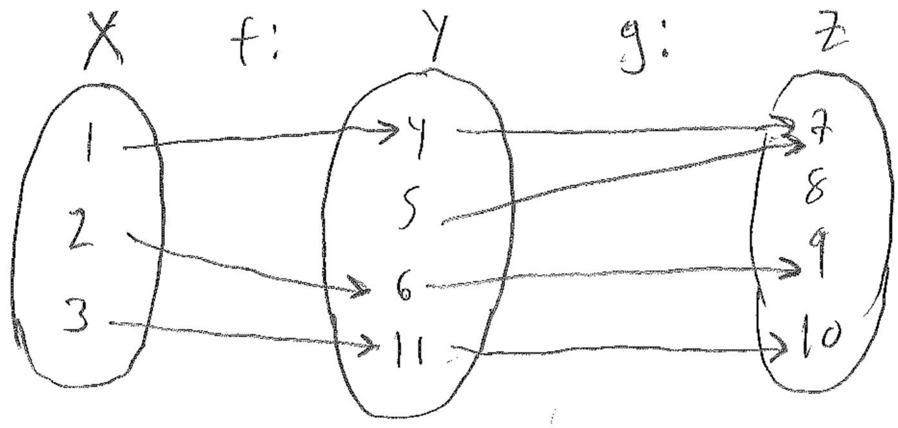
Name:______________________________

5. [20 pts] Let X, Y and Z be sets, and let $f : X \to Y$ and $g : Y \to Z$ be functions, and let $h = g \circ f$ be the composition of f and g. (That is, h is the function from X to Z defined by $h(x) = g(f(x))$.)

(a) [10 pts] Prove that if h is one-to-one, then f is one-to-one as well. [Hint: Assume that $f(x) = f(y)$, and try to use that to prove that $x = y$.]

if f(x) = f(y), then g(f(x)) = g(f(y))
since g(f(x)) = h(x) & h(x) is one-to-one,
& since h(x) = h(y), then x & y must be
the same value since each input into h(x)
maps to a unique output

(b) [10 pts] Give a counterexample to show that it is possible for h to be one-to-one while g is not one-to-one. That is, give examples of sets X, Y and Z and functions $f : X \to Y$ and $g : Y \to Z$ such that $h = g \circ f$ is one-to-one, but g is *not* one-to-one.



in this case, g is not one-to-one since
g(4) = g(5), however, h is since h(1) ≠ h(2) ≠ h(3)
so each h input has a unique output