19F-MATH61-2 Midterm 1

TOTAL POINTS

88 / 100

QUESTION 1

True/False 10 pts

1.1 a **2 / 2**

✓ **- 0 pts** True - Correct

1.2 b **2 / 2**

✓ **- 0 pts** True - Correct

1.3 c **2 / 2**

✓ **- 0 pts** True -Correct

1.4 d **2 / 2**

✓ **- 0 pts** False - Correct

1.5 e **2 / 2**

✓ **- 0 pts** False - Correct

QUESTION 2

Counting 35 pts

2.1 10-card Hands **10 / 10**

✓ **- 0 pts** Correct

2.2 6-letter Strings **10 / 10**

✓ **- 0 pts** Correct

2.3 Permutations of Bookkeeper **3 / 15**

✓ **- 2 pts** C(4, 2) ways first E occurs before first O

✓ **- 5 pts** C(10, 5) spots for Es and Os

✓ **- 5 pts** 5! / 2! ways to place remaining letters

QUESTION 3

Not Equivalence Relations 20 pts

3.1 Subset **5 / 5**

✓ **- 0 pts** Correct

3.2 Distance <1 **7 / 7**

✓ **- 0 pts** Correct

3.3 Nontrivial Intersection **8 / 8**

✓ **- 0 pts** Correct

QUESTION 4

Function Composition 15 pts

4.1 Surjective **7 / 7**

✓ **- 0 pts** Correct

4.2 Injective **8 / 8**

✓ **- 0 pts** Correct

QUESTION 5

5 Induction **20 / 20**

\+ **4 pts** Base Case

\+ **6 pts** Correct goal for inductive step

\+ **5 pts** Almost correctly executed inductive step

\+ **10 pts** Correctly executed inductive step

✓ **+ 20 pts** Correct

\+ **0 pts** Click here to replace this description.

MIDTERM 1 (MATH 61)

MONDAY, OCTOBER 21ST

This exam has 5 (double sided) pages, including the cover page, and a blank page at the end. Please make sure your exam includes each page. Please write your name on *each* page you submit. You will have 50 minutes to complete this exam. **You may not use a calculator**, or consult your textbook, class notes, or any other materials. If you need scratch paper or more space for your answers, please use the extra page at the end.

If there is any work on the blank pages which you would like to have graded, please indicate this CLEARLY on the page for the corresponding problem.

Show your work for these problems, don't just give an answer. If a question asks you to prove something, please write a complete proof. Unless otherwise stated, you may use any results proved in class or in the textbook, but please make it clear when you are doing so. Unless otherwise stated, you will *not* receive full credit for giving the correct answer with no explanation. You may still earn partial credit even if your final answer is incorrect.

Remember that you are bound by a conduct code, and that you may not look at anyone's paper or let anyone look at your paper.

Question	Points	Score
1	10	
2	35	
3	20	
4	15	
5	20	
Total:	100	

1. [10 pts, 2 points each] Mark each of the following statements as either TRUE or FALSE. For this question you do not need to show any work beyond the final answer.

Be sure to read the questions carefully!

(a) The sets $\{3, \clubsuit, \{1,2\}\}$ and $\{\{2,1,1\}, 3, \clubsuit, 3\}$ are equal.	T
(b) There exists a surjective (i.e. onto) function: 16 15 $$f : \mathcal{P}(\{1,2,3,4\}) \to \{A,B,C,D,E\} \times \{X,Y,Z\}$$ where $\mathcal{P}(\{1,2,3,4\})$ is the power set of $\{1,2,3,4\}$.	T
(c) If $X = \{1,2,3,4,5,6\}$ then any injective (i.e. one-to-one) function $f : X \to X$ must also be surjective (i.e. onto). injective :	T
(d) The relation R on $\mathbb{Z}$ defined by xRy if $x \neq y$ is antisymmetric.	F
(e) For any positive integers n and r with $r+1 \leq n$, $C(n,r) < C(n,r+1)$ (i.e. if X is an n-element set, there are always more $(r+1)$-combinations of X than r-combinations of X).	F

2. [35 pts] Compute the following quantities. You may leave your answers in terms of exponents and factorials, but do not leave your final answers in terms of $P(n,r)$ or $C(n,r)$ (so $4^{12}\frac{15!}{3!6!2!}$ would be an acceptable final answer, but $P(10,3)C(18,7)$ would not).

Show your work. It should be clear how you got your answers.

(a) [10 pts] The number of ways to form a 10 card hand from a standard 52 card deck (containing 13 clubs, 13 diamonds, 13 hearts and 13 spades) consisting of exactly 5 clubs, 3 diamonds, 2 hearts and no spades. (The order of the cards in this hand is irrelevant, only the set of 10 cards picked.)

5 from 13 clubs

3 from 13 diamonds

$$\binom{13}{5} \times \binom{13}{3} \times \binom{13}{2} \leftarrow 2 \text{ from } 13 \text{ hearts}$$

$$= \frac{13!}{5!\,8!} \cdot \frac{13!}{10!\,3!} \cdot \frac{13!}{11!\,2!}$$

$$= \frac{(13!)^3}{5!\,8!\,10!\,3!\,11!\,2!}$$

(b) [10 pts] The number of 6 letter strings that can be formed from the letters A, B, C, D, E (allowing repeats) which contain at least one A *and* at least one B. [Hint: It may be easier to count the number of strings which don't satisfy this.]

$\underset{4 \text{ choices}}{\_}\ \overset{4}{\_}\ \overset{4}{\_}\ \overset{4}{\_}\ \overset{4}{\_}\ \overset{4}{\_}$ no A OR no B

no A: 4^6

no B. 4^6

no A and no B: 3^6

no A or no B: $4^6+4^6-3^6$ (inclusion exclusion)

all cases without restricting: 5^6

at least one A and at least one B

= all cases − no A or no B

$= 5^6 - 4^6 - 4^6 + 3^6$

(c) [15 pts] The number of permutations of the letters BOOKKEEPER (that is, strings of length 10 containing exactly 1 B, 2 O's, 2 K's, 3 E's, 1 P and 1 R) *such that the first E occurs before the first O.*

[So 'PREBOKEEOK' would be one such permutation, but 'BOPKEREKOE' would not.]

$\wedge E \wedge O \wedge$ we have a total of 10 letters.

~~First we consider the one where repetition is accepted~~

first, we select out 3 letters. E O O.

These 3 letters must be in the order of $\wedge E \wedge O \wedge O \wedge$

other letters.

The rest of the 7 characters can be wherever they like

we insert them into the slots of $\wedge E \wedge O \wedge O \wedge$.

whenever we insert one char, a slot is created like this

$\wedge E \wedge B \wedge O \wedge O \wedge$

so for the 7 characters, we have $4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10$ choices.

to consider repetition: we have 2K and 3E.

because initial condition E O O only counted the repeated O once,

therefor O doesn't need to be considered here.

So answer is $\dfrac{4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10}{2 \times 3!} = \dfrac{10!}{2 \times (3!)^2}$

3. [20 pts] In each problem below, the relation R is *NOT* an equivalence relation. In each problem, identify a specific property of equivalence relations which fails (either reflexivity, symmetry or transitivity), *and give a specific example to prove that it fails.*

(a) [5 pts] $X = \mathcal{P}(\{1,2,3,4,5\})$, R is the relation on X defined by ARB is $A \subseteq B$ (i.e. A is a subset of B).

fails symmetry

Suppose $A = \{1,2\}$, $B = \{1,2,3\}$.

$A \subseteq B$, $B \nsubseteq A$.

(b) [7 pts] R is the relation on $\mathbb{R}$ (the set of real numbers) defined by xRy if $|x - y| < 1$.

fails transitivity.

suppose $x = 1$, $y = 0.1$, $z = -0.8$

$|x-y| = 0.9$
$|y-z| = 0.9$ $\Rightarrow$ xRy, yRz.

but $|x-z| = 1.8 > 1$ $x \not R z$.

(c) [8 pts] $X = \{A | A \subseteq \{1,2,3,4,5,6\}$ and $|A| = 3\}$ is the set of *three element* subsets of $\{1,2,3,4,5,6\}$. R is the relation on X defined by ARB if $A \cap B \neq \varnothing$.

it fails transitivity.

suppose $X = \{1,2,3\}$, $Y = \{2,3,4\}$, $Z = \{4,5,6\}$,

$X \subseteq U$, $Y \subseteq U$ $Z \subseteq U$

then $X \cap Y = \{2,3\} \neq \varnothing$,

$Y \cap Z = \{4\} \neq \varnothing$ so XRY, YRZ.

but $X \cap Z = \varnothing$, $X \not R Z$.

4. [15 pts] Let X, Y and Z be sets, and let $f : X \to Y$ and $g : Y \to Z$ be functions, and let $h = g \circ f$ be the composition of f and g. (That is, h is the function from X to Z defined by $h(x) = g(f(x))$.)

(a) [7 pts] Prove that if f and g are both onto, then h is onto as well. [Hint: For any $z \in Z$, prove that there is some $x \in X$ with $h(x) = z$.]

~~because g is onto,~~

~~Suppose~~

Because g is onto, for any $z \in Z$, $\exists\, y \in Y$ S.T. $g(y) = z$.

Because f is onto, for any $y \in Y$, $\exists\, x \in X$ S.T. $f(x) = y$.

Therefore, for any $z \in Z$, $\exists\, x \in X$ S.T. $f(x) = y$, $g(y) = z$,

So $g(f(x)) = z$.

because $h(x) = g(f(x))$, ~~h(x) = z~~ substitute into $g(f(x)) = z$

we have $h(x) = z$ for any $z \in Z$.

(b) [8 pts] Prove that if f and g are both one-to-one, then h is one-to-one as well. [Hint: Show that if $h(x_1) = h(x_2)$ for some $x_1, x_2 \in X$ then $x_1 = x_2$.]

if $h(x_1) = h(x_2)$,

then $g(f(x_1)) = g(f(x_2))$

because g is one to one,

$f(x_1) = f(x_2)$

because f is one to one,

$x_1 = x_2$

5. [20 pts] Prove by induction that for any positive integer n,

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{(n-1)^2}+\frac{1}{n^2}\le 2-\frac{1}{n}.$$

Base case: when $n=1$, $LHS = \frac{1}{1^2} = 1$

$RHS = 2-\frac{1}{1} = 1$

$LHS = RHS$.

suppose $\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{(n-1)^2}+\frac{1}{n^2} \le 2-\frac{1}{n}$ is true.

When $n \to n+1$,

we need to prove: $\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{(n-1)^2}+\frac{1}{n^2}+\frac{1}{(n+1)^2} \le 2-\frac{1}{n+1}$

is also true.

Because we can substitute the smaller side with a larger term,

substitute $\frac{1}{1^2}+\cdots+\frac{1}{n^2} \to 2-\frac{1}{n}$ we get that

to prove: $2-\frac{1}{n}+\frac{1}{(n+1)^2} \le 2-\frac{1}{n+1}$

which is to prove: $-\frac{1}{n}+\frac{1}{(n+1)^2} \le -\frac{1}{n+1}$

which is to prove: $\frac{1}{n}-\frac{1}{(n+1)^2} \ge \frac{1}{n+1}$

$LHS = \frac{(n+1)^2-n}{n(n+1)^2} = \frac{n^2+n+1}{n(n+1)^2}$

$RHS = \frac{n(n+1)}{n(n+1)^2} = \frac{n^2+n}{n(n+1)^2}$

because $LHS - RHS = \frac{1}{n(n+1)^2}$, $n \in \mathbb{Z}^+$, LHS is always $+\frac{1}{n(n+1)^2}$ greater than RHS

so, $\frac{1}{n}-\frac{1}{(n+1)^2} \ge \frac{1}{n+1}$

so, $2-\frac{1}{n}+\frac{1}{(n+1)^2} \le 2-\frac{1}{n+1}$

because $2-\frac{1}{n} \ge 1+\cdots+\frac{1}{n^2}$, $1+\cdots+\frac{1}{n^2}+\frac{1}{(n+1)^2} \le 2-\frac{1}{n+1}$ Q.E.D

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