Name:_ ___ ___ ________

1. [10 pts, 2 points each] Mark each of the following statements as either TRUE or FALSE. For this question you do not need to show any work beyond the final answer.

Be sure to read the questions carefully!

1 2 3 4
12 13 14 23 24 34
123 134 234

(a) The sets $\{3, \clubsuit, \{1,2\}\}$ and $\{\{2,1,1\}, 3, \clubsuit, 3\}$ are equal.	TRUE
(b) There exists a surjective (i.e. onto) function: 15 $$f : \mathcal{P}(\{1,2,3,4\}) \to \{A,B,C,D,E\} \times \{X,Y,Z\}$$ where $\mathcal{P}(\{1,2,3,4\})$ is the power set of $\{1,2,3,4\}$.	FALSE
(c) If $X = \{1,2,3,4,5,6\}$ then any injective (i.e. one-to-one) function $f : X \to X$ must also be surjective (i.e. onto).	TRUE
(d) The relation R on $\mathbb{Z}$ defined by xRy if $x \neq y$ is antisymmetric.	FALSE
(e) For any positive integers n and r with $r+1 \leq n$, $C(n,r) < C(n,r+1)$ (i.e. if X is an n-element set, there are always more $(r+1)$-combinations of X than r-combinations of X).	FALSE

$$\frac{n}{r!(n-r)!} < \frac{n}{(r+1)!(n-(r+1))!}$$

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2. [35 pts] Compute the following quantities. You may leave your answers in terms of exponents and factorials, but do not leave your final answers in terms of $P(n,r)$ or $C(n,r)$ (so $4^{12}\frac{15!}{3!6!2!}$ would be an acceptable final answer, but $P(10,3)C(18,7)$ would not).

Show your work. It should be clear how you got your answers.

(a) [10 pts] The number of ways to form a 10 card hand from a standard 52 card deck (containing 13 clubs, 13 diamonds, 13 hearts and 13 spades) consisting of exactly 5 clubs, 3 diamonds, 2 hearts and no spades. (The order of the cards in this hand is irrelevant, only the set of 10 cards picked.)

$$\binom{13}{5}\binom{13}{3}\binom{13}{2}$$

$$= \frac{13!}{5!8!}\cdot\frac{13!}{3!10!}\cdot\frac{13!}{2!11!}$$

$$= \frac{13\cdot12\cdot11\cdot10\cdot9}{5\cdot4\cdot3\cdot2}\cdot\frac{13\cdot12\cdot11}{3\cdot2}\cdot\frac{13\cdot12}{2}$$

$$= 13\cdot11\cdot3\cdot3\cdot13\cdot11\cdot2\cdot13\cdot6$$

$$= 13^3\cdot11^2\cdot6\cdot3^2\cdot2 \text{ ways}$$

(b) [10 pts] The number of 6 letter strings that can be formed from the letters A, B, C, D, E (allowing repeats) which contain at least one A *and* at least one B. [Hint: It may be easier to count the number of strings which don't satisfy this.]

Total strings : 5^5

Strings not containing A and B : 3^5

Strings containing at least one A and at least one B : $5^5 - 3^5$ strings

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(c) [15 pts] The number of permutations of the letters BOOKKEEPER (that is, strings of length 10 containing exactly 1 B, 2 O's, 2 K's, 3 E's, 1 P and 1 R) *such that the first E occurs before the first O.*

[So 'PREBOKEEOK' would be one such permutation, but 'BOPKEREKOE' would not.]

Permutations of BOOKKEEPER: $\dfrac{10!}{2!2!3!}$

E after O = 9!   O E _ _ _ _ _ _ _ _

Total such that first E occurs before first O: $\dfrac{10!}{2!2!3!} - 9!$

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3. [20 pts] In each problem below, the relation R is *NOT* an equivalence relation. In each problem, identify a specific property of equivalence relations which fails (either reflexivity, symmetry or transitivity), *and give a specific example to prove that it fails.*

(a) [5 pts] $X = \mathcal{P}(\{1, 2, 3, 4, 5\})$, R is the relation on X defined by ARB is $A \subseteq B$ (i.e. A is a subset of B).

The symmetry property fails. For example, $A = \{1, 2, 5\}$ and $B = \{1, 2, 3, 5\}$ such that $A, B \in X$ and $A \subseteq B$ so ARB but $B \not\subseteq A$. The symmetry property fails since if ARB it doesn't necessarily follow that BRA.

(b) [7 pts] R is the relation on $\mathbb{R}$ (the set of real numbers) defined by xRy if $|x - y| < 1$.

The transitivity property fails. For example, $x = 0.1$, $y = 0.2$ and $z = 1.15$. such that $x, y, z \in \mathbb{R}$, $|x - y| = 0.1 < 1$, and $|y - z| = 0.95 < 1$ so xRy and yRz. However, this doesn't imply xRz since $|x - z| = 1.05 \not< 1$ so the transitivity property fails.

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(c) [8 pts] $X = \{A | A \subseteq \{1,2,3,4,5,6\} \text{ and } |A| = 3\}$ is the set of *three element* subsets of $\{1,2,3,4,5,6\}$. R is the relation on X defined by ARB if $A \cap B \neq \varnothing$.

The transitive property doesn't hold. For example, $A = \{1,2,3\}$, $B = \{3,4,5\}$, $C = \{4,5,6\}$ such that $A, B, C \in X$ and $A \cap B \neq \varnothing$, $B \cap C \neq \varnothing$ so ARB and BRC. However, this doesn't imply ARC since $A \cap C = \varnothing$ so the transitivity property fails.

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4. [15 pts] Let X, Y and Z be sets, and let $f : X \to Y$ and $g : Y \to Z$ be functions, and let $h = g \circ f$ be the composition of f and g. (That is, h is the function from X to Z defined by $h(x) = g(f(x))$.)

(a) [7 pts] Prove that if f and g are both onto, then h is onto as well. [Hint: For any $z \in Z$, prove that there is some $x \in X$ with $h(x) = z$.]

For any $z \in Z$, there is a $f(x) \in Y$ such that $g(f(x)) = z$ since g is onto. For any $f(x) \in Y$, there is an $x \in X$ which maps to it since f is onto and maps to the codomain of Y. Therefore, there is some $x \in X$ such that $h(x) = z$ and maps from X to the codomain of Z so h is onto.

(b) [8 pts] Prove that if f and g are both one-to-one, then h is one-to-one as well. [Hint: Show that if $h(x_1) = h(x_2)$ for some $x_1, x_2 \in X$ then $x_1 = x_2$.]

If $h(x_1) = h(x_2)$ for some $x_1, x_2 \in X$ then $g(f(x_1)) = g(f(x_2))$.

Since g is one-to-one, $g(f(x_1)) = g(f(x_2))$ only if $f(x_1) = f(x_2)$.

Since f is one-to-one, $f(x_1) = f(x_2)$ only if $x_1 = x_2$.

Therefore, if $h(x_1) = h(x_2)$ for some $x_1, x_2 \in X$ then $x_1 = x_2$ and h is one-to-one.

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5. [20 pts] Prove by induction that for any positive integer n,

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{(n-1)^2} + \frac{1}{n^2} \leq 2 - \frac{1}{n}.$$

Let $P(n)$ be the statement $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{(n-1)^2} + \frac{1}{n^2} \leq 2 - \frac{1}{n}$

Base case: $P(1)$, $1 \leq 2 - \frac{1}{1} = 1$ so true

Assuming $P(n)$, show $P(n) \Rightarrow P(n+1)$

Want to show $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \frac{1}{(n+1)^2} \leq 2 - \frac{1}{n+1}$

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \frac{1}{n^2} + \frac{1}{(n+1)^2} \leq 2 - \frac{1}{n} + \frac{1}{(n+1)^2} \quad \text{using } P(n)$$

$$= \frac{2n-1}{n} + \frac{1}{(n+1)^2}$$

$$= \frac{2n+1}{n+1}$$

$$= 2 - \frac{1}{n+1}$$

Since we have shown $P(n) \Rightarrow P(n+1)$ true by induction.