Midterm 2

Instructions: Please do each question on a separate page, and make sure that you write neatly and clearly so that it shows up on Gradescope. Remember that you may not discuss the exam with other students, or post the exam questions online in any fashion. If you have questions, please submit a question to the instructors on Piazza or email Professor Cameron. The exam is due by 8 am Los Angeles time on Tuesday 2/23. Make sure that you also do the multiple choice questions on Gradescope!

1. In this question, write down your answer, no need for any justification. You can leave your answer in terms of factorials, combination symbols, permutation symbols, etc. Please clearly box your answers in your submission to Gradescope.

   (a) (2 points) How many injective functions are there from $\{1, 2, 3\}$ to $\{1, 2, 3, 4, 5, 6, 7\}$?

   (b) (2 points) How many subsets are there of $\{1, 2, 3, 4, 5, 6, 7\}$ that are the range of an injective function with domain $\{1, 2, 3\}$?

   (c) (2 points) What is the smallest number of people that must be in a group in order to guarantee that at least three people in the group were born in the same month (of possibly different years)? (The answer is not three; if you have three people they could all be born in the same month, but this is not *guaranteed*).

   (d) (2 points) What is the coefficient of a^5b^2 in $(a + b)^7$?

   (e) (2 points) What is the coefficient of a^5b^2c in $(a + b + c)^8$?

2. Let t_n be the number of strings of length n in $\{a, b, c\}$ that don't have aa as a substring.

   (a) (1 point) What are t_0 and t_1?

   (b) (3 points) Find a recurrence relation for t_n.

   (c) (2 points) Show that the number of as that appear in a length n string in $\{a, b, c\}$ that doesn't have aa as a substring is at most $\lfloor \frac{n+1}{2} \rfloor$.

   (d) (4 points) Show that $t_n = \sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} 2^{n-i} \binom{n-i+1}{i}$.

3. (a) (7 points) Show that if G is a simple graph with n vertices (where n is a positive integer) and each vertex has degree greater than or equal to $\frac{n-1}{2}$, then the diameter of G is 2 or less.

   (b) (3 points) If G is a (not necessarily simple) graph with n vertices where each vertex has degree greater than or equal to $\frac{n-1}{2}$, is the diameter of G necessarily 2 or less? Either prove that the answer to this question is "yes" or give a counterexample.

4. (a) (6 points) Show that if G is a connected graph and e is an edge in G that is part of a cycle, then the graph G' obtained from G by deleting the edge e (and keeping all other edges and vertices) is also connected.

   (b) (4 points) Show that removing one edge from a connected graph where all vertices have even degree results in a graph that is connected.