Math 61-1 Midterm 2 version a

[redacted]

TOTAL POINTS

43 / 50

QUESTION 1

Multiple choice 10 pts

1.1 2 / 2

✓ **- 0 pts Correct (d)**

- 2 pts Incorrect

1.2 2 / 2

✓ **- 0 pts Correct (a)**

- 2 pts Incorrect

1.3 2 / 2

✓ **- 0 pts Correct (c)**

- 2 pts Incorrect

- 2 pts No Answer

1.4 2 / 2

✓ **- 0 pts Correct (b)**

- 2 pts Incorrect

1.5 2 / 2

✓ **- 0 pts Correct (d)**

- 2 pts Incorrect

- 2 pts No Answer

QUESTION 2

Short answer 10 pts

2.1 2 / 2

✓ **- 0 pts Correct (176)**

- 2 pts Incorrect

- 2 pts No answer

2.2 2 / 2

✓ **- 0 pts Correct (34650)**

- 2 pts Incorrect

2.3 0 / 2

- 0 pts Correct (21)

✓ **- 2 pts Incorrect**

- 2 pts No answer

2.4 2 / 2

✓ **- 0 pts Correct (960)**

- 2 pts Incorrect

- 0.5 pts Gave term, not just coefficient

- 1.5 pts Didn't multiply by 8 (120) or similar

- 2 pts No answer

2.5 2 / 2

✓ **- 0 pts Correct (302400)**

- 2 pts Incorrect

- 2 pts No answer

QUESTION 3

Euler paths 10 pts

3.1 criteria for euler path 5 / 5

✓ **+ 5 pts Correct**

+ 2 pts Euler cycle criterion

+ 1 pts Euler cycle criterion (missing connected, or other mistake)

+ 2 pts Reduction to graph with even degrees

+ 1 pts Reduction to graph with even degrees (with mistake)

+ 1 pts Correct explanation of how to get Euler path from Euler cycle

+ 0 pts Incorrect

3.2 application of euler paths 6 / 5

✓ **- 0 pts Correct**

✓ **+ 1 pts Click here to replace this description.**

+ 2 pts Click here to replace this description.

QUESTION 4

4 Pigeon hole **10 / 10**

✓ **- 0 pts** **Correct**

- 3 pts Incorrect Partition

- 3 pts No Pigeonhole

- 2 pts Minor error

- 10 pts Blank

QUESTION 5

Hypercube 10 pts

5.1 recurrence for edges **0 / 4**

- 0 pts Correct

✓ **- 4 pts** **empty**

- 2 pts large mistake

- 2 pts you are assuming the desired conclusion

- 1 pts need to explain how the hypercube it built out of smaller ones

- 3 pts can't just do examples

- 1 pts incomplete

- 3 pts I don't see how this shows that the recurrence is true

5.2 number of edges **4 / 6**

- 0 pts Correct

- 1 pts Need to use induction to show that formula is true

- 4 pts wrong answer, need to use iteration

- 6 pts empty

- 3 pts wrong answer, this is why you need to use induction to show that answer is true

- 4 pts incomplete

✓ **- 2 pts** **wrong answer**

your 2^k -1 should be a k instead.

gradescope

Midterm 2

Name: [redacted]

Student ID: [redacted]

Section:

Tuesday:	Thursday:	
1A	1B	TA: Albert Zheng
1C	1D	TA: Benjamin Spitz
1E	1F	TA: Eilon Reisin-Tzur

Instructions: Do not open this exam until instructed to do so. You will have 50 minutes to complete the exam. Please print your name and student ID number above, and circle the number of your discussion section. **You may not use calculators,** books, notes, or any other material to help you. Please make sure **your phone is silenced** and stowed where you cannot see it. Remember that you are bound by a conduct code, and that you may not look at anyone's paper or let anyone look at your paper, nor may you work on the exam after it has ended. There is some scratch paper at the back of the exam, if you do work there indicate so on the relevant question. Do not write on the backs of the pages. Please circle or box your final answers.

Please get out your id and be ready to show it when you turn in your exam.

Please do not write below this line.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

1. (10 points) Circle the correct answer (only one answer is correct for each question)

   1. K_n has an Euler cycle:
      (a) For every n
      (b) For no n
      (c) For n even
      (d) For n odd

   2. If X is a set with 10 elements and Y is a set with 3 elements and $f : X \to Y$:
      (a) there are at least 4 distinct elements of X, x_1, x_2, x_3, x_4 with $f(x_1) = f(x_2) = f(x_3) = f(x_4)$.
      (b) it is possible that there are *not* 4 distinct elements of X, x_1, x_2, x_3, x_4 with $f(x_1) = f(x_2) = f(x_3) = f(x_4)$.
      (c) there are at least 5 distinct elements of X, x_1, x_2, x_3, x_4, x_5 with $f(x_1) = f(x_2) = f(x_3) = f(x_4) = f(x_5)$.
      (d) f could be one-to-one

$3 \times 3 + 1$

   3. If $s_0 = 2$ and $s_1 = 1$ and for $n \geq 2$ $s_n = s_{n-1} + s_{n-2}$ then $s_n =$
      (a) $\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n$
      (b) $\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$
      (c) $\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$
      (d) none of the above

Question 1 continues on the next page...

Question 1 continued...

4. The number of ways of arranging a math book, a computer science book, and an English book on a shelf is:

history book as well

   (a) 4

   (b) 4!

4 books

   (c) $1 + 2 + 3 + 4$

   (d) $4 - 3 + 2 - 1$

5. The number of relations on a set with n elements that are both symmetric and reflexive is:

   (a) 2^{n^2}

$2^{n^2} - n$

   (b) $2^{n^2-n} + 2^{\frac{n^2+n}{2}} - 2^{\frac{n^2-n}{2}}$

   (c) $2^{n^2} + 2^{\frac{n^2+n}{2}} - 2^{\frac{n^2-n}{2}}$

   (d) none of the above

2

n^2-n

$\underline{2^{n^2} - n}$ $+$

n^2-n

2. In this question write down your answer, no need for any justification. Leave your answers in a form involving factorials, $P(n, m)$, $\binom{n}{m}$, exponents, etc.

   (a) (2 points) What is the number of solutions to the equation $x_1 + x_2 + x_3 = 20$ where $x_3 \leq 10$ and x_1, x_2, x_3 are nonnegative integers?

   $\binom{22}{2} - \binom{11}{2}$

   (b) (2 points) What is the number of ways of rearranging the letters of the word "MISSISSIPPI"?

   $\frac{11!}{4!4!2!}$

   (c) (2 points) What is the number of binary strings of length 6 that don't contain 11 as a substring?

   $3 \times 2 \times 1 = 3!$

   (d) (2 points) What is the coefficient of x^7y^3 in the polynomial $(x+2y)^{10}$?

   $2^3\binom{10}{3}$

   (e) (2 points) I have 10 distinct books. Two are by Ernest Hemingway and three are by Virginia Woolf. How many ways are there to arrange these books on a shelf if the ones by Hemingway must be next to each other and the ones by Woolf must not be next to each other?

   \- - - - - - - - - -

   $\boxed{2 \times 6! \times P(7,3)}$

3. (a) (5 points) Show that if a connected graph has exactly 2 vertices of odd degree, then there is a path in the graph that visits each edge exactly once. (Feel free to use what we know about Euler cycles from lecture)

If a graph has exactly 2 vertices ~~o~~ of odd degree, then ~~the~~ all the other vertices have even degree.

Construct an edge 'e' between the two vertices of odd degree, say v_1 and v_2. $\therefore$ v_1 and v_2 now have an even degree.

Since, all the vertices now have an even degree in this new graph say G', $\therefore$ there is an euler cycle from v_1 to v_1, such that $v_1 \ldots v_2 e v_1$.

If there is an Euler cycle from v_1, then there must be a path from v_1 to v_2 and then from v_2 to v_1 through e that visits all the edges.

$\therefore$ There is an Euler path from v_1 to v_2

$\Rightarrow$ There is a path from v_1 to v_2 in the original graph which visits each edge exactly once.

Question 3 continues on the next page...

(b) (5 points) A house has a front door and a back door and no other doors leading outside. Inside each room of the house, there are (exactly) two doors that each go to other rooms of the house [1] (and the only way to enter or leave a room is through a door). Each room can be reached from the front door. Show that it is possible to go in through the front door and leave through the back door while going through each door in the house exactly once. Feel free to use the previous part of this problem.

The above problem can be modeled as a graph.

~~Ever~~ The house is the graph where every door is a vertex and each room is an edge.

Since, the ~~back~~ front door opens into a room but has no room on its other side (since it will be outside the house), therefore it has an odd degree. Similarly, for the back door. E.2

Every other door in the house, is between two different rooms and thus has a degree of 2 which is even. ~~8~~ And since all the rooms are connected, the graph is connected. ★

~~&~~ Since, the front door (vertex) and back door (vertex) have an odd degree and every other door (vertex) has an even degree, ∴ There is an euler path from the front door to the ~~g~~ back door.

That is, there is a path from the front door to the back door that visits each room exactly once.

★ The do-
rooms just
after and
before the
front and
back door
have degr
4.

[1] To clarify the rooms just inside the front and the back doors have three doors, one of them is the front/back door, and the other two are the doorways leading to other rooms of the house

4. (10 points) Suppose that six distinct integers are selected from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Prove that at least two of these six integers sum to 11.

We can partition the above set as follows,

$\{1, 10\}$, $\{2, 9\}$, $\{3, 8\}$, $\{4, 7\}$, $\{5, 6\}$

There are 5 such partitions.

If we take any 6 numbers and assign them to these partitions, there will be at least one partition with 2 numbers by the Pigeonhole principle.

The sum of the numbers in each partition is 11.

Since, ~~there~~ at least one partition will have 2 numbers that sum up to 11,

therefore, there are at least two of the six chosen integers which sum to up to 11.

5. Recall the n-dimensional hypercube. This is a graph with vertices labeled by binary strings of length n, with an edge between two vertices if they differ in exactly one digit. Let c_n be the number of edges in the n-dimensional hypercube.

   (a) (4 points) Show that c_n satisfies the recurrence $c_n = 2^{n-1} + 2c_{n-1}$

Question 5 continues on the next page...

Question 5 continued...

(b) (6 points) Solve your recurrence from the previous part of this question to find a formula for c_n.

$c_n = 2^{n-1} + 2c_{n-1}$

$c_n = 2^{n-1} + 2(2^{n-1} + 2c_{n-2})$

$\Rightarrow c_n = 2^{n-1} + 2^n + 2^2 c_{n-2}$

$\Rightarrow c_n = 2^{n-1} + 2^n + 2^2(2^{n-1} + 2c_{n-2})$

$\Rightarrow c_n = 2^{n-1} + 2^n + 2^{n+1} + 2^3 c_{n-3}$

$a^1 \quad ar^{n-1}$

$\vdots$

$\cancel{n+k}$

$n-1+n-2+k$

$\Rightarrow c_n = 2^{n-1} + 2^n + \dots + 2^{n-2+k} + 2^k c_{n-k}$

$\Rightarrow c_n = \dfrac{2^{n-1}(2^k - 1)}{2-1} + 2^k c_{n-k}$ $\left[\because a + ar + ar^2 + \dots + ar^{n-1} = \dfrac{a(r^n - 1)}{r-1}\right]$

$\Rightarrow \boxed{c_n = 2^{n-1}(2^k - 1) + 2^k(c_{n-k})}$

Let's prove our guess using induction ~~for~~ on k.

$\cancel{(c_{n-1})2 + 2^{n-1}}$

~~For n=1, k=0~~

~~$c_1 = 1$ edge [∵ n=1 cube: 0 — 1]~~

~~And RHS: 2^{1-1} ($c_n = 2^{n-1}(2-1) + 2^0(c_{n-0})$~~

~~$\Rightarrow c_n = 2^{n-1} + c_n$~~

Base case:

$k = 1$

$c_n = 2^{n-1}(2-1) + 2(c_{n-1}) = 2^{n-1} + 2c_{n-1}$

which is given formula.

$\therefore$ The base case holds.

Induction step: Let $c_n = 2^{n-1}(2^m - 1) + 2^m(c_{n-m})$ be true for any m between 1 and ~~all~~

~~$1 \le k$~~ $1 \le m \le k$

continued on next page

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$x_1 + x_2 + x_3 = 20$

$0 \le x_3 \le 10$

$x_1 \ge 0 \quad x_2 \ge 0$

$x_3 \ge 11$

$\binom{20+3-1}{3-1}$

$20 - 11 = 9$

$\binom{9+3-1}{3-1} = \binom{11}{2}$

$\binom{22}{2}$

$2 + 2^2 + 2^2 C_{n-2}$

$2 + 2^2 + 2^3 + 2^3 C_{n-3}$

$a + ar + ar^2 + \dots + ar^{n-1}$

$2 + 2^2 + 2^3 + \dots + 2^k + 2^k (\dots)$

$2 + 2^2 + 2^3 + \dots$

$n - 1 = k \Rightarrow n = k+1$

$\frac{10 \quad 00 \quad 01}{3 \times 2 \times 1}$

2^{n-1} $\frac{2(2^{k+1}-1)}{2-1}$

8 $\quad$ 2 $\quad 2^{n-1}$

$\binom{10}{2} \times 2$

8

_H _1 _2 _3 _4 _5 _

P(

6!