Math 61-1 Midterm 2 version a

[redacted]

TOTAL POINTS

49 / 50

QUESTION 1

Multiple choice 10 pts

1.1 **2 / 2**

✓ **- 0 pts Correct (d)**

- 2 pts Incorrect

1.2 **2 / 2**

✓ **- 0 pts Correct (a)**

- 2 pts Incorrect

1.3 **2 / 2**

✓ **- 0 pts Correct (c)**

- 2 pts Incorrect

- 2 pts No Answer

1.4 **2 / 2**

✓ **- 0 pts Correct (b)**

- 2 pts Incorrect

1.5 **2 / 2**

✓ **- 0 pts Correct (d)**

- 2 pts Incorrect

- 2 pts No Answer

QUESTION 2

Short answer 10 pts

2.1 **2 / 2**

✓ **- 0 pts Correct (176)**

- 2 pts Incorrect

- 2 pts No answer

2.2 **2 / 2**

✓ **- 0 pts Correct (34650)**

- 2 pts Incorrect

2.3 **1 / 2**

- 0 pts Correct (21)

- 2 pts Incorrect

- 2 pts No answer

- 1 Point adjustment

Close (there are 4 possibilities with three 1's)

2.4 **2 / 2**

✓ **- 0 pts Correct (960)**

- 2 pts Incorrect

- 0.5 pts Gave term, not just coefficient

- 1.5 pts Didn't multiply by 8 (120) or similar

- 2 pts No answer

2.5 **2 / 2**

✓ **- 0 pts Correct (302400)**

- 2 pts Incorrect

- 2 pts No answer

QUESTION 3

Euler paths 10 pts

3.1 criteria for euler path **5 / 5**

✓ **+ 5 pts Correct**

+ 2 pts Euler cycle criterion

+ 1 pts Euler cycle criterion (missing connected, or other mistake)

+ 2 pts Reduction to graph with even degrees

+ 1 pts Reduction to graph with even degrees (with mistake)

+ 1 pts Correct explanation of how to get Euler path from Euler cycle

+ 0 pts Incorrect

3.2 application of euler paths **6 / 5**

✓ **- 0 pts** **Correct**

✓ **+ 1 pts** **Click here to replace this description.**

+ 2 pts Click here to replace this description.

QUESTION 4

4 Pigeon hole 10 / 10

✓ **- 0 pts** **Correct**

- 3 pts Incorrect Partition

- 3 pts No Pigeonhole

- 2 pts Minor error

- 10 pts Blank

QUESTION 5

Hypercube 10 pts

5.1 recurrence for edges 4 / 4

✓ **- 0 pts** **Correct**

- 4 pts empty

- 2 pts large mistake

- 2 pts you are assuming the desired conclusion

- 1 pts need to explain how the hypercube it built out of smaller ones

- 3 pts can't just do examples

- 1 pts incomplete

- 3 pts I don't see how this shows that the recurrence is true

5.2 number of edges 5 / 6

- 0 pts Correct

✓ **- 1 pts** **Need to use induction to show that formula is true**

- 4 pts wrong answer, need to use iteration

- 6 pts empty

- 3 pts wrong answer, this is why you need to use induction to show that answer is true

- 4 pts incomplete

- 2 pts wrong answer

J. Cameron Math 61 Mon, February 25th, 2019

Midterm 2

Name: [redacted]

Student ID: [redacted]

Section:

	Tuesday:	Thursday:	
	1A	1B	TA: Albert Zheng
	1C	1D	TA: Benjamin Spitz
	1E	1F	TA: Eilon Reisin-Tzur

Instructions: Do not open this exam until instructed to do so. You will have 50 minutes to complete the exam. Please print your name and student ID number above, and circle the number of your discussion section. **You may not use calculators,** books, notes, or any other material to help you. Please make sure **your phone is silenced** and stowed where you cannot see it. Remember that you are bound by a conduct code, and that you may not look at anyone's paper or let anyone look at your paper, nor may you work on the exam after it has ended. There is some scratch paper at the back of the exam, if you do work there indicate so on the relevant question. Do not write on the backs of the pages. Please circle or box your final answers.

Please get out your id and be ready to show it when you turn in your exam.

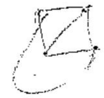
Please do not write below this line.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

1. (10 points) Circle the correct answer (only one answer is correct for each question)

   1. K_n has an Euler cycle:

      (a) For every n

      (b) For no n

      (c) For n even

      (d) For n odd

      $\lceil \frac{10}{3} \rceil = 4$

   2. If X is a set with 10 elements and Y is a set with 3 elements and $f : X \to Y$:

      (a) there are at least 4 distinct elements of X, x_1, x_2, x_3, x_4 with $f(x_1) = f(x_2) = f(x_3) = f(x_4)$.

      (b) it is possible that there are *not* 4 distinct elements of X, x_1, x_2, x_3, x_4 with $f(x_1) = f(x_2) = f(x_3) = f(x_4)$.

      ~~(c)~~ there are at least 5 distinct elements of X, x_1, x_2, x_3, x_4, x_5 with $f(x_1) = f(x_2) = f(x_3) = f(x_4) = f(x_5)$.

      ~~(d)~~ f could be one-to-one

   3. If $s_0 = 2$ and $s_1 = 1$ and for $n \geq 2$ $s_n = s_{n-1} + s_{n-2}$ then $s_n =$

      (a) $\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n$

      (b) $\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$

      (c) $\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$

      (d) none of the above

      $\frac{1+\cancel{\sqrt{5}}+1-\cancel{\sqrt{5}}}{2} = \frac{2}{2} = 1$

Question 1 continues on the next page...

Question 1 continued...

4. The number of ways of arranging a math book, a computer science book, and an English book on a shelf is:

   (a) 4

   (b) 4!

   (c) $1 + 2 + 3 + 4$

   (d) $4 - 3 + 2 - 1$

History

4 total

5. The number of relations on a set with n elements that are both symmetric and reflexive is:

   (a) 2^{n^2}

   (b) $2^{n^2-n} + 2^{\frac{n^2+n}{2}} - 2^{\frac{n^2-n}{2}}$

   4 + 8 - 2

   (c) $2^{n^2} + 2^{\frac{n^2+n}{2}} - 2^{\frac{n^2-n}{2}}$

   (d) none of the above

a b c d e

(a, a)

(b, b)

2^3

2^6

$n=2 \rightarrow 2^1$

$n=3 \rightarrow 2^3$

$n=4 \rightarrow 2^6$

$n=5 \rightarrow 2^{10}$

$2^{\frac{n(n-1)}{2}}$

2. In this question write down your answer, no need for any justification. Leave your answers in a form involving factorials, $P(n,m)$, $\binom{n}{m}$, exponents, etc.

(a) (2 points) What is the number of solutions to the equation $x_1+x_2+x_3=20$ where $x_3 \leq 10$ and x_1, x_2, x_3 are nonnegative integers?

$\binom{20+3-1}{3-1} = \binom{22}{2}$ total, if $x_3 \geq 11$, $\binom{9+3-1}{3-1} = \binom{11}{2}$

$\boxed{\binom{22}{2} - \binom{11}{2}}$

(b) (2 points) What is the number of ways of rearranging the letters of the word "MISSISSIPPI"?

$\frac{11!}{4!\,4!\,2!}$

(c) (2 points) What is the number of binary strings of length 6 that don't contain 11 as a substring?

$\sum_{i=0}^{3} \binom{6-i+1}{i}$

$\boxed{\binom{6}{6} + \binom{6}{1} + 10 + 2}$

no 1's, one 1, 2 1's, 3 1's

(d) (2 points) What is the coefficient of x^7y^3 in the polynomial $(x+2y)^{10}$?

$2^3 \binom{10}{7}$

(e) (2 points) I have 10 distinct books. Two are by Ernest Hemingway and three are by Virginia Woolf. How many ways are there to arrange these books on a shelf if the ones by Hemingway must be next to each other and the ones by Woolf must not be next to each other?

$6!\,2! \times \binom{7}{3} \times 3!$

3. (a) (5 points) Show that if a connected graph has exactly 2 vertices of odd degree, then there is a path in the graph that visits each edge exactly once. (Feel free to use what we know about Euler cycles from lecture)

If we insert an edge between the vertices w/ odd degree, then every vertex will have even degree and the graph will still be connected. Thus the graph will have an Euler cycle. Choose the starting point of the cycle to be one of the vertices with odd degree, and let the last edge we traverse in the cycle be the edge we inserted. Thus, the last vertex we visit before returning to the starting point is the 2^{nd} vertex that originally had odd degree. Thus, if we remove the inserted edge, we will have a path that visits each edge in the graph exactly once.

Question 3 continues on the next page...

(b) (5 points) A house has a front door and a back door and no other doors leading outside. Inside each room of the house, there are (exactly) two doors that each go to other rooms of the house [1] (and the only way to enter or leave a room is through a door). Each room can be reached from the front door. Show that it is possible to go in through the front door and leave through the back door while going through each door in the house exactly once. Feel free to use the previous part of this problem.

If we model this as a graph with rooms as vertices and doors as graphs, then the front room and back room each have degree 3 and every other room has degree 2. Thus, we can apply our results from part a) since exactly 2 ~~vertices~~ vertices have odd degree & graph is connected (we can reach every ~~room~~ room from front door). This graph must have a path traversing each edge exactly once, and we know it starts/ends at the odd degree vertices. Thus, we start at the front door, use every door once, and leave from back door.

[1] To clarify the rooms just inside the front and the back doors have three doors, one of them is the front/back door, and the other two are the doorways leading to other rooms of the house

4. (10 points) Suppose that six distinct integers are selected from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Prove that at least two of these six integers sum to 11.

Pigeonhole Principle

The subsets $\{1,10\}$, $\{2,9\}$, $\{3,8\}$, $\{4,7\}$, and $\{5,6\}$ all add up to 11. We have 5 subsets.

If the 5 subsets are pigeonholes and the six integers we select are pigeons, then there are more pigeons than pigeonholes, so at least 2 ~~numbers~~ integers we select must be in the same subset. Since each subset adds up to 11, the 2 ~~numbers~~ integers we pick in the same subset must add up to 11, so at least 2 of our 6 integers sum to 11.

5. Recall the n-dimensional hypercube. This is a graph with vertices labeled by binary strings of length n, with an edge between two vertices if they differ in exactly one digit. Let c_n be the number of edges in the n-dimensional hypercube.

(a) (4 points) Show that c_n satisfies the recurrence $c_n = 2^{n-1} + 2c_{n-1}$

We can construct an n-cube by taking an n-1 -cube, duplicating it, and adding 0's to the front of the strings of the original n-1 -cube & 1's to the front of the strings of the copied n-1 -cube. This gives us $2c_{n-1}$ edges. Then, connect the vertices of the 2 n-1 -cubes if they differ by 1 digit. Corresponding vertices of the n-1 -cubes will be connected. An n-1 -cube has 2^{n-1} vertices, so this adds 2^{n-1} edges. Thus, an n-cube has $c_n = 2^{n-1} + 2c_{n-1}$ edges.

Question 5 continues on the next page...

Question 5 continued...

(b) (6 points) Solve your recurrence from the previous part of this question to find a formula for c_n.

$C_n = 2^{n-1} + 2C_{n-1}$

$C_n = 2^{n-1} + 2(2^{n-2} + 2C_{n-2}) = 2^{n-1} + 2^{n-1} + 2^2 C_{n-2}$

$C_n = 2^{n-1} + 2(2^{n-2} + 2(2^{n-3} + 2C_{n-3})) = 2^{n-1} + 2^{n-1} + 2^{n-1} + 2^3 C_{n-3}$

$C_n = k \times 2^{n-1} + 2^k C_{n-k}$ $\longrightarrow$ $C_1 = 1$

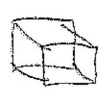
let $k = n-1$

$C_n = (n-1)2^{n-1} + 2^{n-1} C_1$

$C_n = n 2^{n-1} (1)$ $\quad$ $C_n = n 2^{n-1}$

$C_1 = 1$ ✓

$C_2 =$  4 ✓

$C_3 =$  12 ✓