Math 61-1 Midterm 2 version b

EUGENE LO

TOTAL POINTS

44 / 50

QUESTION 1

Multiple choice 10 pts

1.1 **2 / 2**

✓ **- 0 pts** **Correct (d)**

- 2 pts Incorrect

1.2 **0 / 2**

- 0 pts Correct (d)

- 2 pts Incorrect

✓ **- 2 pts** **No Answer**

1.3 **2 / 2**

✓ **- 0 pts** **Correct (b)**

- 2 pts Incorrect

1.4 **0 / 2**

- 0 pts Correct (c)

✓ **- 2 pts** **Incorrect**

1.5 **2 / 2**

✓ **- 0 pts** **Correct (b)**

- 2 pts Incorrect

QUESTION 2

Short answer 10 pts

2.1 **2 / 2**

✓ **- 0 pts** **Correct (960)**

- 2 pts Incorrect

- 0.5 pts Gave term, not just coefficient

- 1.5 pts Didn't multiply by 8 (120) or similar

2.2 **2 / 2**

✓ **- 0 pts** **Correct (34650)**

- 2 pts Incorrect

2.3 **2 / 2**

✓ **- 0 pts** **Correct (302400)**

- 2 pts Incorrect

- 2 pts No answer

2.4 **2 / 2**

✓ **- 0 pts** **Correct (21)**

- 2 pts Incorrect

- 2 pts No answer

2.5 **2 / 2**

✓ **- 0 pts** **Correct (176)**

- 2 pts Incorrect

- 2 pts No answer

QUESTION 3

Euler paths 10 pts

3.1 criteria for euler paths **4 / 5**

+ 5 pts Correct

✓ **+ 2 pts** **Euler cycle criterion**

+ 1 pts Euler cycle criterion (missing connected or mistake)

✓ **+ 2 pts** **Reduction to even degree graph**

+ 1 pts Reduction to even degree graph (with mistake)

+ 1 pts Correct explanation of how to get Euler path from Euler cycle

+ 0 pts Incorrect

Unclear how you constructed a path from the given Euler cycle.

3.2 application of euler paths **6 / 5**

✓ **- 0 pts** **Correct**

+ 2 pts Click here to replace this description.

+ **1 pts** Click here to replace this description.

+ **1** **Point adjustment**

QUESTION 4

4 pigeon hole 10 / 10

✓ **- 0 pts** **Correct**

- 3 pts Wrong partition

- 3 pts No Pigeonhole

- 2 pts Minor error

- 10 pts Blank

QUESTION 5

hypercube 10 pts

5.1 recurrence for edges 4 / 4

✓ **- 0 pts** **Correct**

- 4 pts empty

- 1 pts Sounds like you are saying the n-1 dim hypercube has n-1 vertices

- 1 pts need to explain how the hypercube is built out of smaller ones

- 4 pts empty

- 0.5 pts How do you get the recurrence?

- 2 pts large mistake

- 2 pts you are assuming the desired conclusion

- 3 pts can't just do examples

You have double counted all the edges, but this is almost an argument for why there are n2^{n-1} edges.

5.2 formula for edges 4 / 6

- 0 pts Correct

- 4 pts wrong answer, need to use iteration

- 1 pts need to use induction to show that answer is true

- 3 pts wrong answer, this is why you need to use induction to show answer is true

- 6 pts empty

- 4 pts incomplete

✓ **- 2 pts** **wrong answer**

Midterm 2

Name: Eugene Lo

Student ID: 905108982

Section:

	Tuesday:	Thursday:	
	1A	1B	TA: Albert Zheng
	1C	1D	TA: Benjamin Spitz
	1E	1F	TA: Eilon Reisin-Tzur

Instructions: Do not open this exam until instructed to do so. You will have 50 minutes to complete the exam. Please print your name and student ID number above, and circle the number of your discussion section. **You may not use calculators,** books, notes, or any other material to help you. Please make sure **your phone is silenced** and stowed where you cannot see it. Remember that you are bound by a conduct code, and that you may not look at anyone's paper or let anyone look at your paper, nor may you work on the exam after it has ended. There is some scratch paper at the back of the exam, if you do work there indicate so on the relevant question. Do not write on the backs of the pages. Please circle or box your final answers.

Please get out your id and be ready to show it when you turn in your exam.

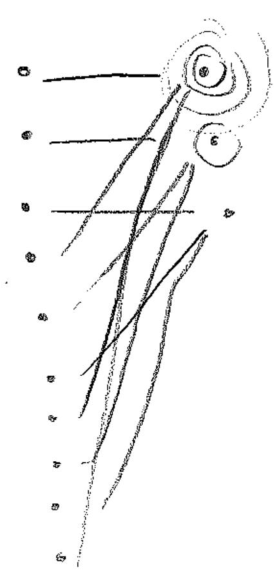
Please do not write below this line.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

1. (10 points) Circle the correct answer (only one answer is correct for each question)

1. If X is a set with 10 elements and Y is a set with 3 elements and $f : X \to Y$:
   (a) f could be one-to-one
   (b) it is possible that there are *not* 4 distinct elements of X, x_1, x_2, x_3, x_4 with $f(x_1) = f(x_2) = f(x_3) = f(x_4)$.
   (c) there are at least 5 distinct elements of X, x_1, x_2, x_3, x_4, x_5 with $f(x_1) = f(x_2) = f(x_3) = f(x_4) = f(x_5)$.
   (d) there are at least 4 distinct elements of X, x_1, x_2, x_3, x_4 with $f(x_1) = f(x_2) = f(x_3) = f(x_4)$.

2. The number of relations on a set with n elements that are both symmetric and reflexive is:
   (a) $2^{n^2} + 2^{\frac{n^2+n}{2}} - 2^{\frac{n^2-n}{2}}$
   (b) $\underline{2^{n^2-n}} + 2^{\frac{n^2+n}{2}} - 2^{\frac{n^2-n}{2}}$
   (c) 2^{n^2} reflexive
   (d) none of the above



n(n-1)

3. K_n has an Euler cycle:
   (a) For no n
   (b) For n odd
   (c) For n even
   (d) For every n

Question 1 continues on the next page...

Question 1 continued...

4. If $s_0 = 2$ and $s_1 = 1$ and for $n \geq 2$ $s_n = s_{n-1} + s_{n-2}$ then $s_n =$
   (a) $\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$
   (b) $\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n$
   (c) $\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$
   (d) none of the above

5. The number of ways of arranging a math book, a computer science book, and an English book on a shelf is:
   history book
   (a) $4 - 3 + 2 - 1$
   (b) $4!$
   (c) $1 + 2 + 3 + 4$
   (d) 4

2. In this question write down your answer, no need for any justification. Leave your answers in a form involving factorials, $P(n,m)$, $\binom{n}{m}$, exponents, etc.

(a) (2 points) What is the coefficient of x^7y^3 in the polynomial $(x+2y)^{10}$?

$(x+2y)^{10}$ $\binom{10}{3}(x)^7(2y)^3$

$\boxed{\binom{10}{3}\cdot 8}$

(b) (2 points) What is the number of ways of rearranging the letters of the word "MISSISSIPPI"?

$\boxed{\frac{11!}{4!4!2!}}$

(c) (2 points) I have 10 distinct books. Two are by Ernest Hemingway and three are by Virginia Woolf. How many ways are there to arrange these books on a shelf if the ones by Hemingway must be next to each other and the ones by Woolf must not be next to each other?

$\frac{7!}{5!}$

$\frac{7\times6\times5\times4!}{4!}$

$\boxed{2!\times 6!\times P(7,3)}$

H H

$10-3=7-2+1=6$ 7 B 6 B 5 B _ B _ B _ B _

(d) (2 points) What is the number of binary strings of length 6 that don't contain 11 as a substring?

$\frac{7!}{4!}$

$\frac{7\times6\times5\times\cancel{4!}}{\cancel{4!}}$

$S_n = S_{n-1} + S_{n-2}$ 1 1 _ _ _ _ $\boxed{21}$

$S_0=0$ $S_1=2$ $S_2=3$ $S_3=5$ $S_4=8$ $S_5=13$ $S_6=21$

(e) (2 points) What is the number of solutions to the equation $x_1+x_2+x_3=20$ where $x_3 \leq 10$ and x_1,x_2,x_3 are nonnegative integers?

total: $\binom{20+3-1}{3-1} = \binom{22}{2}$

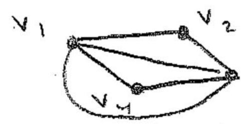
$x \geq 11$: $\binom{(20-11)+3-1}{3-1} = \binom{11}{2}$

$\boxed{\binom{22}{2}-\binom{11}{2}}$

$x_1 \geq 0$ $x_2 \geq 0$, $0 \leq x_3 \leq 10$

3. (a) (5 points) Show that if a connected graph has exactly 2 vertices of odd degree, then there is a path in the graph that visits each edge exactly once. (Feel free to use what we know about Euler cycles from lecture)

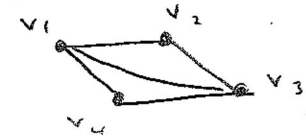
From lecture, we know that if we have a connected graph G with all vertices having an even degree, there is an Euler cycle that visits every edge once, because every time you leave on an edge incident on a vertex, you can come back on another edge incident on that vertex. For example:



has an Euler cycle from v_1 back to v_1 because each vertex has an even degree.

Now, let's say we have a new connected graph G with exactly 2 vertices of odd degree. For example:

G



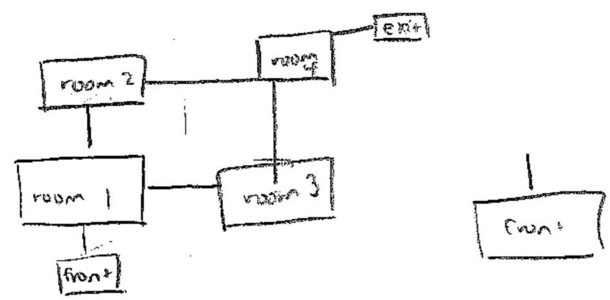
v_1 and v_3 both have an odd degree of 3.

We can construct an Euler cycle by making a new graph G' that adds an edge e incident on v_1 and v_3, making them even.

Because they are all even there is an Euler cycle and a path P from v_1 to v_1. This means that in our graph G the edges can be partitioned to a cycle from V_1 to V_1 and a path P' from v_1 to some v_n (in our case v_3). Because the only edge missing to make it a path P on G' is e, which takes it from v_3 to v_1, this means that our cycle and P' partitions visited every edge once in graph G. Thus, it is a path that visits each edge once.

Question 3 continues on the next page...

(b) (5 points) A house has a front door and a back door and no other doors leading outside. Inside each room of the house, there are (exactly) two doors that each go to other rooms of the house [1] (and the only way to enter or leave a room is through a door). Each room can be reached from the front door. Show that it is possible to go in through the front door and leave through the back door while going through each door in the house exactly once. Feel free to use the previous part of this problem.



[1] To clarify the rooms just inside the front and the back doors have three doors, one of them is the front/back door, and the other two are the doorways leading to other rooms of the house

4. (10 points) Suppose that six distinct integers are selected from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Prove that at least two of these six integers sum to 11.

Six distinct integers are selected from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. We can group the numbers into pairs that sum up to 11.

$\{1, 10\}$, $\{2, 9\}$, $\{3, 8\}$, $\{4, 7\}$, $\{5, 6\}$. There are five such pairs.

If I select six integers from the set, and each integer I pick is distinct, then that means that there are more numbers picked (pigeons) than there are pairs (holes). As a result, by the Pigeonhole principle, there must be at least 2 integers that are in the same pair, which sum to 11. Thus, at least 2 of the six integers sum to eleven.

5. Recall the n-dimensional hypercube. This is a graph with vertices labeled by binary strings of length n, with an edge between two vertices if they differ in exactly one digit. Let c_n be the number of edges in the n-dimensional hypercube.

(a) (4 points) Show that c_n satisfies the recurrence $c_n = 2^{n-1} + 2c_{n-1}$

C_1 : # of edges in a 1-cube = 1 

C_2 : # of edges in a 2-cube = 4 

# of edges in an n-cube : There are 2^n vertices in an n-cube. Each one of these vertices has a degree of $n-1$, so the total # of edges is $2^n(n-1)$.

What this recursion is doing is finding the # of vertices in the previous $(n-1)$-cube and adding the new edges formed $(2C_{n-1})$ because an edge is forme between two vertices if they differ in one digit : one vertex from one $n-1$ cube to one vertex in the other $n-1$ cube.

Question 5 continued...

(b) (6 points) Solve your recurrence from the previous part of this question to find a formula for c_n.

$C_n = 2^{n-1} + 2C_{n-1}$

$C_{n-1} = 2^{n-2} + 2C_{n-2}$

$C_n = 2^{n-1} + 2(2^{n-2} + 2C_{n-2})$

$C_n = 2^{n-1} + 2^{n-1} + 2^2 C_{n-2}$

$C_n = 2 \cdot 2^{n-1} + 2^2 C_{n-2}$

$C_{n-2} = 2^{n-3} + 2C_{n-3}$

$C_n = 2 \cdot 2^{n-1} + 2^2(2^{n-3} + 2C_{n-3})$

$C_n = 2 \cdot 2^{n-1} + 2^{n-1} + 2^3 C_{n-3}$

$C_n = 2^2 \cdot 2^{n-1} + 2^3 C_{n-3}$

$C_n = 2^{n-k+1} \cdot 2^{n-1} + 2^k C_{n-k-1}$

Induction: Suppose for k $C_n = 2^{n-k} \cdot 2^{n-1} + 2^{k-1} C_{n-k-1}$

prove $k+1$ $C_n = 2^{n-(k+1)+1} + 2^{n-1} + 2^{k+1} C_{n-k-1}$

$= 2(2^{n-(k+1)} + 2^{n-1} + 2^k C_{n-k-1})$

$C_n = 2^{n-k+1} \cdot 2^{n-1} + 2^k C_{n-k}$ if $k = n-1$

$C_n = 2^2 \cdot 2^{n-1} + 2^{n-1} C_1$ $n-(k+1)$

$= 2^n + 2^{n-1} C_1$ $C_1 = 1$ $n-k$

$C_n = 2^n + 2^{n-1}$

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