Midterm 2

Name: [redacted]

Student ID: [redacted]

Section:

	Tuesday:	Thursday:	
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	1C	1D	TA: Benjamin Spitz
	1E	1F	TA: Eilon Reisin-Tzur

Instructions: Do not open this exam until instructed to do so. You will have 50 minutes to complete the exam. Please print your name and student ID number above, and circle the number of your discussion section. **You may not use calculators,** books, notes, or any other material to help you. Please make sure **your phone is silenced** and stowed where you cannot see it. Remember that you are bound by a conduct code, and that you may not look at anyone's paper or let anyone look at your paper, nor may you work on the exam after it has ended. There is some scratch paper at the back of the exam, if you do work there indicate so on the relevant question. Do not write on the backs of the pages. Please circle or box your final answers.

Please get out your id and be ready to show it when you turn in your exam.

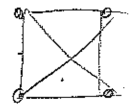
Please do not write below this line.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

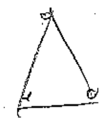
1. (10 points) Circle the correct answer (only one answer is correct for each question)

1. K_n has an Euler cycle:

degree = 3



degree 2



   (a) For every n
   (b) For no n
   (c) For n even
   (d) For n odd

2. If X is a set with 10 elements and Y is a set with 3 elements and $f : X \to Y$:
   (a) there are at least 4 distinct elements of X, x_1, x_2, x_3, x_4 with $f(x_1) = f(x_2) = f(x_3) = f(x_4)$.
   (b) it is possible that there are *not* 4 distinct elements of X, x_1, x_2, x_3, x_4 with $f(x_1) = f(x_2) = f(x_3) = f(x_4)$.
   (c) there are at least 5 distinct elements of X, x_1, x_2, x_3, x_4, x_5 with $f(x_1) = f(x_2) = f(x_3) = f(x_4) = f(x_5)$.
   (d) ~~f could be one-to-one~~

3. If $s_0 = 2$ and $s_1 = 1$ and for $n \geq 2$ $s_n = s_{n-1} + s_{n-2}$ then $s_n =$
   (a) $\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n$
   (b) $\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$ $\longrightarrow$ $S_0 = 1$ $S_1 = 1$
   (c) $\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$
   (d) none of the above

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)$$

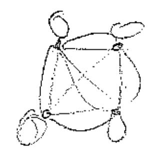
$$\frac{\frac{1}{\sqrt{5}} + 1}{2} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{1+\sqrt{5}}{2\sqrt{5}}$$

Question 1 continues on the next page...

Question 1 continued...

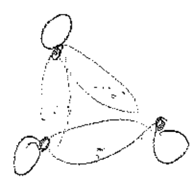
4. The number of ways of arranging a math book, a computer science book, and an English book on a shelf is:
   (a) 4 & history 4 books
   (b) 4!
   (c) $1+2+3+4$
   (d) $4-3+2-1$

5. The number of relations on a set with n elements that are both symmetric and reflexive is:
   (a) $\cancel{2^{n^2}}$
   (b) $2^{n^2-n} + 2^{\frac{n^2+n}{2}} - 2^{\frac{n^2-n}{2}}$
   (c) $2^{n^2} + 2^{\frac{n^2+n}{2}} - 2^{\frac{n^2-n}{2}}$
   (d) none of the above



2^6

4



$2^3 = 8$

$n=3$

$2^6 + 2^6 - 2^3$ $\quad 2^{12}$

2. In this question write down your answer, no need for any justification. Leave your answers in a form involving factorials, $P(n, m)$, $\binom{n}{m}$, exponents, etc.

(a) (2 points) What is the number of solutions to the equation $x_1 + x_2 + x_3 = 20$ where $x_3 \leq 10$ and x_1, x_2, x_3 are nonnegative integers?

_ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ 11

$$\binom{21}{1} + \binom{20}{1} + \binom{19}{1} + \binom{18}{1} + \binom{17}{1} + \binom{16}{1} + \binom{15}{1} + \binom{14}{1} + \binom{13}{1} + \binom{12}{1} + \binom{11}{1}$$

0 1 2 3 4 5 6 7 8 9 10

(b) (2 points) What is the number of ways of rearranging the letters of the word "MISSISSIPPI"?

$$\frac{11!}{4!\,4!\,2!}$$

(c) (2 points) What is the number of binary strings of length 6 that don't contain 11 as a substring?

-0- 0-0 -0 -0- 6C1
-0-0-0-0- 5C2
-0-0-0- 4C3
-0-0-

$$\binom{6}{1} + \binom{5}{2} + \binom{4}{3}$$

(d) (2 points) What is the coefficient of x^7y^3 in the polynomial $(x+2y)^{10}$?

$$\binom{10}{7} \cdot 2^3$$

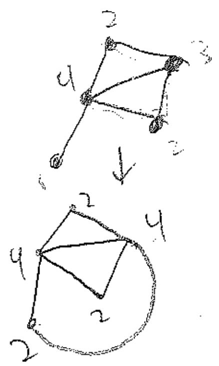
(e) (2 points) I have 10 distinct books. Two are by Ernest Hemingway and three are by Virginia Woolf. How many ways are there to arrange these books on a shelf if the ones by Hemingway must be next to each other and the ones by Woolf must not be next to each other?

$$\binom{7}{3}$$

Basically have 9 books
3 wolfs, 6 others

_ O _ O _ O _ O _ O _ O _

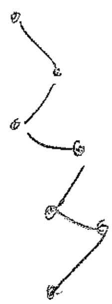
3. (a) (5 points) Show that if a connected graph has exactly 2 vertices of odd degree, then there is a path in the graph that visits each edge exactly once. (Feel free to use what we know about Euler cycles from lecture)



If the graph has exactly 2 vertices of odd degree then there exists a graph with all even degree vertices that is the previous graph but with one extra edge between the two vertices of odd degree. I.E. → example on left. This graph with one extra edge has an ~~exist~~ Euler cycle because all vertices have even degree. If we were to travel said Euler cycle starting & ending at one of the vertices with odd degree and leaving the last path as between those two vertices then we would be able to remove this last step and create a path that travels each edge exactly once & visits all vertices. We know such an Euler cycle exists as described above because the graph is connected so all vertices have nonzero degree. We would always be starting and ending our paths at the odd vertices for this new path that doesn't traverse the new edge.

Question 3 continues on the next page...

(b) (5 points) A house has a front door and a back door and no other doors leading outside. Inside each room of the house, there are (exactly) two doors that each go to other rooms of the house [1] (and the only way to enter or leave a room is through a door). Each room can be reached from the front door. Show that it is possible to go in through the front door and leave through the back door while going through each door in the house exactly once. Feel free to use the previous part of this problem.



Each room is an edge and each
each door is a vertex.

Lets say we connect the front and back door by a room we call "outside". This would be the edge between our two places with odd degree.

Now, every single vertex has degree 2.

This means there is an Euler cycle because there is an even degree to each vertex!

[1]To clarify the rooms just inside the front and the back doors have three doors, one of them is the front/back door, and the other two are the doorways leading to other rooms of the house

4. (10 points) Suppose that six distinct integers are selected from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Prove that at least two of these six integers sum to 11.

$5 + 6 = 11$

$4 + 7 = 11$

$3 + 8 = 11$

$2 + 9 = 11$

$1 + 10 = 11$

Every number ≤ 5 has a corresponding number which is ≥ 5 that you can sum it with and get 11. If we are choosing 6 numbers, then we are guaranteed to get at least one full sum because there are only 5 sets of sums and we are choosing 6 numbers. This is true by the pigeonhole principle.

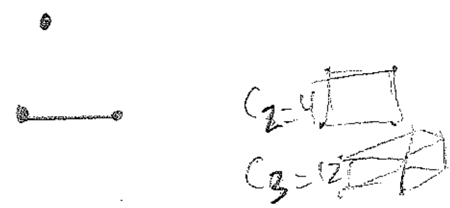
You can pick 5 numbers without summing to 11 but the 6th number guarantees that we have at least 2 numbers that pair to sum to 11.

5. Recall the n-dimensional hypercube. This is a graph with vertices labeled by binary strings of length n, with an edge between two vertices if they differ in exactly one digit. Let c_n be the number of edges in the n-dimensional hypercube.

(a) (4 points) Show that c_n satisfies the recurrence $c_n = 2^{n-1} + 2c_{n-1}$

$C_0 = 0$

$C_1 = 1$



Base case = $C_2 = 2^0 + 2C_0$

$= 1$ ✓

Inductive Step = Assume $C_n = 2^{n-1} + 2C_{n-1}$

C_{n+1} = For every edge with entries $a_1 \dots a_n$ $(a_1, \dots, a_n)$ There now exists two extra edges $(a_1, \dots, a_n, 1)$ \$ $(a_1, \dots, a_n,$ giving us 2^n extra edges, also for previously defined edges that differ by 1. Such as $i_1, \dots, i_n$ $j_1, \dots, j_n$, there exists two extra edges

$(i_1, \dots, i_n, 0)$ $\big\rangle$ differ by 1
$(j_1, \dots, j_n, 0)$

$(i_1, \dots, i_n, 1)$ $\big\rangle$ differ by 1
$(j_1, \dots, j_n, 1)$

This is $2 \cdot C_n$ edges

so we have

$C_{n+1} = 2^n + 2C_n$

Question 5 continues on the next page...

Question 5 continued...

(b) (6 points) Solve your recurrence from the previous part of this question to find a formula for c_n.

$C_0 = 0$

$C_1 = 1$ $\quad C_3 = 12$

$C_2 = 4$

$C_n = 2^{n-1} + 2C_{n-1}$

$C_n = \lambda^2$

~~$C_n = 2^{3(n-1)}$~~

$\lambda^2 = 2^{n-1} + 2\lambda$

$\lambda^2 - 2\lambda - 2^{n-1} = 0$

$\lambda = \dfrac{2 \pm \sqrt{4 - 4(2^{n-1})}}{2}$

$\lambda = 1 \pm \sqrt{1 - 2^{n-1}}$

$C_n = a_1\left(1 + \sqrt{1-2^{n-1}}\right)^n + a_2\left(1 - \sqrt{1-2^{n-1}}\right)^n$

$C_1 = a_1(1 + \sqrt{1-}$

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