Midterm 2

Name:

Student ID:

Section:

Instructions: Do not open this exam until instructed to do so. You will have 50 minutes to complete the exam. Please print your name and student ID number above, and circle the number of your discussion section. **You may not use calculators,** books, notes, or any other material to help you. Please make sure **your phone is silenced** and stowed where you cannot see it. Remember that you are bound by a conduct code, and that you may not look at anyone's paper or let anyone look at your paper, nor may you work on the exam after it has ended. There is some scratch paper at the back of the exam, if you do work there indicate so on the relevant question. Do not write on the backs of the pages. Please circle or box your final answers.
Please get out your id and be ready to show it when you turn in your exam.

Please do not write below this line.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

1. (10 points) Circle the correct answer (only one answer is correct for each question)

   1. K_n has an Euler cycle:
      (a) For every n
      (b) For no n  
      (c) For n even
      (d) For n odd

   2. If X is a set with 10 elements and Y is a set with 3 elements and $f : X \to Y$:
      (a) there are at least 4 distinct elements of X, x_1, x_2, x_3, x_4 with $f(x_1) = f(x_2) = f(x_3) = f(x_4)$.
      (b) it is possible that there are *not* 4 distinct elements of X, x_1, x_2, x_3, x_4 with $f(x_1) = f(x_2) = f(x_3) = f(x_4)$.
      (c) there are at least 5 distinct elements of X, x_1, x_2, x_3, x_4, x_5 with $f(x_1) = f(x_2) = f(x_3) = f(x_4) = f(x_5)$.
      (d) f could be one-to-one

X_1
X_2
X_3

   3. If $s_0 = 2$ and $s_1 = 1$ and for $n \geq 2$ $s_n = s_{n-1} + s_{n-2}$ then $s_n =$
      (a) $\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n$
      (b) $\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$
      (c) $\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$
      (d) none of the above

$S_2 = S_1 + S_0 = 3$

$\frac{1+2\sqrt{5}+5}{4}$ $\frac{1-2\sqrt{5}+5}{4}$

$\frac{3+\sqrt{5}}{2}$ $\frac{3-\sqrt{5}}{2}$

Question 1 continues on the next page...

Question 1 continued...

4. The number of ways of arranging a math book, a computer science book, and an English book on a shelf is:

4 books history book

   (a) 4
   (b) 4!
   (c) $1+2+3+4$
   (d) $4-3+2-1$

5. The number of relations on a set with n elements that are both symmetric and reflexive is:
   (a) 2^{n^2}
   (b) $2^{n^2-n}+2^{\frac{n^2+n}{2}}-2^{\frac{n^2-n}{2}}$
   (c) $2^{n^2}+2^{\frac{n^2+n}{2}}-2^{\frac{n^2-n}{2}}$
   (d) none of the above

$\{1,2,3,4\}$

1R1 2R2 3R3 4R4

$\{1,2\}$

$\emptyset$ 2

2. In this question write down your answer, no need for any justification. Leave your answers in a form involving factorials, $P(n,m)$, $\binom{n}{m}$, exponents, etc.

(a) (2 points) What is the number of solutions to the equation $x_1+x_2+x_3=20$ where $x_3 \le 10$ and x_1, x_2, x_3 are nonnegative integers?

$\binom{20+3-1}{3-1} - \binom{9+3-1}{3-1} = \binom{22}{2} - \binom{11}{2}$

(b) (2 points) What is the number of ways of rearranging the letters of the word "MISSISSIPPI"?

1 2 3 4 5 6 7 8 9 10 11

$\frac{11!}{4!4!2!}$

(c) (2 points) What is the number of binary strings of length 6 that don't contain 11 as a substring?

01 10 00

1 2 3 4 5 6

2 3 5 8 13 21

$f_n = f_{n-1} + f_{n-2}$

$\boxed{21}$

(d) (2 points) What is the coefficient of x^7y^3 in the polynomial $(x+2y)^{10}$?

$\binom{10}{3} \cdot (2)^3 = 8\binom{10}{3}$

(e) (2 points) I have 10 distinct books. Two are by Ernest Hemingway and three are by Virginia Woolf. How many ways are there to arrange these books on a shelf if the ones by Hemingway must be next to each other and the ones by Woolf must not be next to each other?

$5! \cdot P(6,1) \cdot P(7,3)$

3. (a) (5 points) Show that if a connected graph has exactly 2 vertices of odd degree, then there is a path in the graph that visits each edge exactly once. (Feel free to use what we know about Euler cycles from lecture)

A graph contains an Euler cycle if and only if the graph is connected and each vertex has even degree. Suppose G is a connected graph with 2 vertices of odd degree. If we add an edge in G between the two vertices of odd degree, then each vertex has even degree, so there exists an Euler cycle. Call this new graph G'. The section of the Euler cycle of G' in G beginning at one the vertices which ~~originally had odd~~ · has odd degree in G forms an Euler path in G, ~~which Euler path~~ ~~w~~ or a path which visits each edge exactly once.

Question 3 continues on the next page...

(b) (5 points) A house has a front door and a back door and no other doors leading outside. Inside each room of the house, there are (exactly) two doors that each go to other rooms of the house [1] (and the only way to enter or leave a room is through a door). Each room can be reached from the front door. Show that it is possible to go in through the front door and leave through the back door while going through each door in the house exactly once. Feel free to use the previous part of this problem.

We can create a graph G with ~~edges between~~ rooms of the house being vertices and edges between connected rooms. We see that this graph has two vertices of odd degree (the rooms with the front and back doors). Since each room has two doors which can each go to other rooms, each other vertex has even degree. This graph is connected since each room can be reached from the front door. Thus, ~~since this~~ since this is a connected graph with exactly two vertices of odd degree, there is a path through each door in the house exactly once.

[1] To clarify the rooms just inside the front and the back doors have three doors, one of them is the front/back door, and the other two are the doorways leading to other rooms of the house

4. (10 points) Suppose that six distinct integers are selected from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Prove that at least two of these six integers sum to 11.

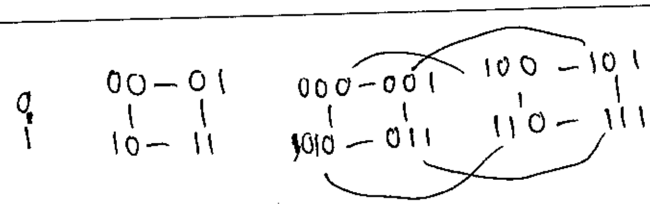
$\{(1,10), (2,9), (3,8), (4,7), (5,6)\}$

Suppose the 5 pairs of numbers listed above are the pigeonholes. Since there are 6 distinct integers (pigeons), by the Pigeonhole Principle, there must be one pigeonhole with two integers. Since the two numbers in a pair sum to 11, there must be at least two of the six integers that sum to 11.

5. Recall the n-dimensional hypercube. This is a graph with vertices labeled by binary strings of length n, with an edge between two vertices if they differ in exactly one digit. Let c_n be the number of edges in the n-dimensional hypercube.

$+2C_{n-1}$

(a) (4 points) Show that c_n satisfies the recurrence $c_n = 2^{n-1} \cancel{+2c_n}$.



We can see that for each added dimension, the number of vertices doubles, since we can add either a 0 or a 1 in front of each binary string. For each binary string, if we add a 0 or 1, the resulting strings have an edge between them, since they differ only by the first digit. There are 2^{n-1} of these edges, since we can form 2^{n-1} binary strings, then add 0 and 1.

We can also see that if we add a 0 to each binary string of length $n-1$, then the edges in graph of $n-1$ still exist in the graph of n. This also goes for if we add a 1 to each binary string of length $n-1$. ~~Thus,~~ There are $2C_{n-1}$ of these edges.

Thus, the number of edges in an n-dim hypercube is the sum of these, or $2^{n-1} + 2C_{n-1}$.

Question 5 continues on the next page...

Question 5 continued...

(b) (6 points) Solve your recurrence from the previous part of this question to find a formula for c_n.

$$c_n = 2^{n-1} + 2c_{n-1}$$

$$(2^{n-2} + 2c_{n-2}) = 2 \cdot 2^{n-1} + 2^2 c_{n-2}$$

$$= 2^{n-[illegible]} + 2(\ldots)$$

$$= 2 \cdot 2^{n-[illegible]} + 2^2 (2^{n-3} + 2c_{n-3})$$

$$= 3 \cdot 2^{n-1} + 2^3 c_{n-3}$$

$$c_0 = 0 \quad c_1 = 1$$

$$c_n = n(2^{n-1}) + 2^n c_0$$

$$= n(2^{n-1})$$

This page has been left intentionally blank. You may use it as scratch paper. It will not be graded unless indicated very clearly here and next to the relevant question.