Midterm 1

Instructions: Please do each question on a separate page, and make sure that you write neatly and clearly so that it shows up on Gradescope. Remember that you may not discuss the exam with other students, or post the exam questions online in any fashion. If you have questions, please submit a question to the instructors on Piazza or email Professor Cameron. The exam is due by 8 am Los Angeles time on Tuesday 1/26. Make sure that you also do the multiple choice questions on Gradescope!

1. In this question write down your answer, no need for any justification. Please clearly box your answers in your submission to Gradescope.

   (a) (2 points) How many relations are there on $X = \{a, b, c, d, e\}$ that are symmetric, antisymmetric, and transitive?

   (b) (2 points) Consider the equivalence relation F on the integers, defined by $F = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x - y = 4k \text{ for some } k \in \mathbb{Z}\}$. Give a set consisting of exactly one element from each equivalence class of F (so the cardinality of your set should be the cardinality of the set of equivalence classes of F, and each equivalence class should be represented in your set).

   (c) (2 points) Give two symmetric relations S, R on $\{a, b, c\}$ such that $S \circ R$ is not symmetric.

   (d) (2 points) How many functions are there from the set $\{a, b, c, d, e\}$ to the set $\{0, 1\}$?

   (e) (2 points) Give sets X and Y and functions $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ is the identity function on X (i.e. for all $x \in X$, $g \circ f(x) = x$), but $f \circ g$ is *not* the identity function on Y.

2. For X a set let $\mathcal{O}(X) = \{A \in \mathcal{P}(X) : |A| \text{ is odd}\}$ and let $\mathcal{E}(X) = \{A \in \mathcal{P}(X) : |A| \text{ is even}\}$.

   (a) (2 points) Give the sets $\mathcal{O}(\{a, b, c\})$ and $\mathcal{E}(\{a, b, c\})$ (list out the elements of each set).

   (b) (8 points) Show that for $|X|$ finite and $|X| \geq 1$,

$$|\mathcal{O}(X)| = |\mathcal{E}(X)| = 2^{|X|-1}.$$

   Note: There was a similar problem on your homework (that you did not turn in). If want to use that problem from your homework you need to include a solution. You are free to use anything we did in class, including information about $|\mathcal{P}(X)|$.

3. Suppose that R is an equivalence relation on X, and that $f : X \to Y$ is a function. Let $\mathcal{E}$ be the set of equivalence classes of R, so $\mathcal{E} = \{[x]_R : x \in X\}$. Define a relation h from $\mathcal{E}$ to Y by

$$h = \{(A, y) \in \mathcal{E} \times Y : \text{ there exists } x \in A \text{ with } f(x) = y\}.$$

   (a) (4 points) Let $X = \{0, 1, 2, 3\}$ and $Y = \{a, b, c\}$, and let $f : X \to Y$ be defined by $f(0) = a$, $f(1) = a$, $f(2) = b$, $f(3) = b$ and let R be the relation on X where $R = \{(x, y) \in X \times X : x - y \text{ is even}\}$.

   In this example, list $\mathcal{E}$ and h. List the elements of these sets with no repetitions.

   (b) (6 points) Now, show (in general, not just for the example from the first part of this problem) that if h is a function then for all $x, y \in X$, if $(x, y) \in R$ then $f(x) = f(y)$.

4. Suppose that $f : X \to Y$ and $g : Y \to Z$ are functions.

   (a) (5 points) Show that if $g \circ f$ is onto, then g is onto.

   (b) (5 points) Show that if $g \circ f$ is one-to-one, then f is one-to-one.