1. (10 points) Circle the correct answer (only one answer is correct for each question)

   1. The relation on the integers defined by xRy if $x - y$ is odd is:
      (a) an equivalence relation
      (b) symmetric, but not transitive or reflexive
      (c) reflexive, but not symmetric or transitive
      (d) symmetric and transitive, but not reflexive

   2. If $f : X \to Y$ is an one-to-one function and $g : Y \to Z$ is an onto function, then $g \circ f$ is necessarily:
      (a) one-to-one
      (b) onto
      (c) a bijection
      (d) not necessarily one-to-one or onto

   3. If $f : X \to Y$ and $g : Y \to Z$ are functions and $g \circ f$ is a bijection, then:
      (a) f must be one-to-one and g must be onto
      (b) f and g must both be bijections
      (c) f must be onto
      (d) g must in one-to-one

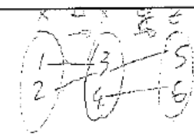
   4. The function $f : \{a, b\}^* \times \{a, b\}^* \to \{a, b\}^*$ (X^* is the set of strings in X) defined by $f((\alpha, \beta)) = \alpha\beta$ is:
      (a) onto and one-to-one
      (b) one-to-one but not onto
      (c) neither one-to-one nor onto
      (d) onto but not one-to-one

   5. For X a set with n elements, how many relations are there on X that are both partial orders and equivalence relations?
      (a) more than 1 and less than n
      (b) more than n
      (c) exactly n
      (d) exactly 1

2. In this question write down your answer, no need for any justification.

   (a) (2 points) Give an example of a functions $f : X \to Y$, $g : Y \to Z$ so that g is onto but $g \circ f$ is not onto.

   $X = \{1,2\}$ $Y = \{3,4\}$ $Z = \{5,6\}$
   $f = \{(1,3),(2,3)\}$ $\quad g \circ f = \{(1,5),(1,5)\}$
   $g = \{(3,5),(4,6)\}$

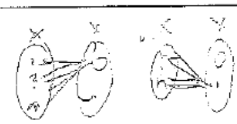
   (b) (2 points) What is $\sum_{i=0}^{100} i$? Feel free to write your answer as the product or sum of a few numbers.

   $\sum_{i=0}^{100} i = 0+1+2+3+\cdots 100$
   $= \frac{(100+1)(100)}{2} = 101 \cdot 50 = \boxed{5050}$

   (c) (2 points) List elements of the equivalence relation on $\{a, b, c, d, e\}$ determined by the partition $\{\{a,b\},\{c\},\{d,e\}\}$.

   $R = \{(a,a),(a,b),(b,a),(b,b),(c,c),(d,d),(d,e),$
   $(e,d),(e,e)\}$

   (d) (2 points) If X is a set with n elements where $n \geq 1$, how many onto functions are there from X to $\{0,1\}$?

   $2^n - 2$

   (e) (2 points) Give an example of a relation on $X = \{a, b, c\}$ that is reflexive and symmetric but not transitive.

   $R = \{(a,a),(b,b),(c,c),(a,b),(b,a),(b,c),(c,b)\}$

3. Let $X = \mathbb{Z} \times \mathbb{Z}_{>0}$ (i.e. ordered pairs of integers where the second integer is positive), and define a relation Q on X by $(a,b)Q(c,d)$ if $ad = bc$.

(a) (8 points) Show that Q is an equivalence relation on X.

For $(a,b) \in X$, $ab = ab$ so $(a,b)Q(a,b)$. Q is reflexive.

Suppose $(a,b)Q(c,d)$ then $ab = cd$. Then $cd = ab$, so $(c,d)Q(a,b)$. Q is symmetric.

Suppose $(a,b)Q(c,d)$ and $(c,d)Q(f,g)$, then $ab = cd$ and $cd = fg$. So, $ab = cd = fg$. Then $(a,b)Q(f,g)$ so Q is transitive.

Since Q is reflexive, symmetric, and transitive, Q is an equivalence relation on X.

(b) (2 points) Give three different elements of $[(2,3)]_Q$.

$(3,2)$, $(1,6)$, $(6,1)$

4. (10 points) Recall that the Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$ is defined by $F_0 = 0, F_1 = 1$, and for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$. Show that for $n \geq 0$, $\sum_{i=0}^{n} F_i = F_{n+2} - 1$.

Base Case: For $n=0$, $\sum_{i=0}^{0} F_i = F_0 = 0 = F_2 - 1 = 1 - 1 = 0$

$F_2 = F_1 + F_0 = 1$ Base case holds.

Assume that $\sum_{i=0}^{k} F_i = F_{k+2} - 1$ for some $k \geq 2$.

We want to show that $\sum_{i=0}^{k+1} F_i = F_{k+3} - 1$.

$$\sum_{i=0}^{k+1} F_i = \sum_{i=0}^{k} F_i + F_{k+1}$$

$$= (F_{k+2} - 1) + F_{k+1}$$ by the inductive hypothesis.

$$= (F_{k+2} + F_{k+1}) - 1$$ $F_n = F_{n-1} + F_{n-2}$, $F_{k+3} = F_{k+2} + F_{k+1}$

$$= F_{k+3} - 1$$

Thus, $\sum_{i=0}^{n} F_i = F_{n+2} - 1$ for all $n \geq 0$ by induction.

5. Let X be a set, and let $P = \{(A, B) \in \mathcal{P}(X) \times \mathcal{P}(X) : A \subseteq B\}$ be the set of ordered pairs (A, B) of subsets of X where $A \subseteq B$. Let $\{0, 1, 2\}^X$ denote the set of functions from X to $\{0, 1, 2\}$.

Define a function $F : P \to \{0, 1, 2\}^X$ by for $(A, B) \in P$ the function $F((A, B)) : X \to \{0, 1, 2\}$ is defined by for $x \in X$,

$$F((A,B))(x) = \begin{cases} 0 & x \in A \\ 1 & x \in B \text{ and } x \notin A \\ 2 & x \notin B \end{cases}.$$

(a) (2 points) Let $X = \{a, b, c, d\}$. What is $F((\{a, d\}, \{a, b, d\}))$? Remember your answer should be a function $X \to \{0, 1, 2\}$.

$$F((\{a,d\}, \{a,b,d\}))(x) = \begin{cases} 0 & x \in \{a,d\} \\ 1 & x \in \{a,b,d\} \text{ and } x \notin \{a,d\} \\ 2 & x \notin \{a,b,d\} \end{cases}$$

$= \{(a,0), (b,1), (c,2), (d,0)\}$

(b) (2 points) Again let $X = \{a, b, c, d\}$. Find a pair (A, B) of subsets of X with $A \subseteq B$ so that $F((A, B))$ is the function $g : X \to \{0, 1, 2\}$ defined by $g(a) = 1, g(b) = 1, g(c) = 2, g(d) = 2$.

$g(a) = 1$ $\quad$ $g(b) = 1$ $\quad$ $g(c) = 2$ $\quad$ $g(d) = 2$

$a \in B$ and $a \notin A$ $\quad$ $b \in B$ and $\notin A$ $\quad$ $c \notin B$ $\quad$ $d \notin B$

$(A, B) = (\{ \quad \}, \{a, b \})$

Question 5 continued...

(c) (6 points) Show that for any set X the function $F : P \to \{0, 1, 2\}^X$ defined on the previous page is a bijection.

<u>One-to-one</u>

For (A, B) and $(C, D) \in P$, suppose

$F((A,B)) = F((C,D))$, then the functions

f and g must be the same.

That means, the ordered pairs in f and in g are the same, So the elements that are since the definition of f and g makes the pairs distinct in A and C must be the same, and the ones in B and D must be the same, according to the function produced. Thus, F is one-to-one since $\forall\ (A,B), (C,D) \in P$, if $F((A,B)) = F((C,D))$ then $(A,B) = (C,D)$.

<u>Onto</u>

For a function $g \in \{0,1,2\}^X$, there always exists a pair of set A and B, $(A,B) \in P$ such that $F((A,B)) = g$ since g's definitions depends on $X, A,$ and B and A and B are subsets of X, so any g can be mapped to by F using some $(A,B) \in P$

Since F is one-to-one and onto, F is a bijection.