Math 61-1 Midterm 1 version b

MATTHEW WANG

TOTAL POINTS

41 / 50

QUESTION 1

Multiple choice 10 pts

1.1 2 / 2

✓ **+ 2 pts Correct (b)**

+ **0 pts** Incorrect

1.2 0 / 2

+ **2 pts** Correct (d)

✓ **+ 0 pts Incorrect**

1.3 0 / 2

+ **2 pts** Correct (a)

✓ **+ 0 pts Incorrect**

1.4 2 / 2

✓ **+ 2 pts Correct (d)**

+ **0 pts** Incorrect

1.5 2 / 2

✓ **+ 2 pts Correct (d)**

+ **0 pts** Incorrect

QUESTION 2

Short answer 10 pts

2.1 2 / 2

✓ **- 0 pts Correct**

2.2 2 / 2

✓ **+ 2 pts Correct (5050)**

+ **1 pts** 1+2+...+100

+ **1 pts** n(n+1)/2

+ **0 pts** Incorrect

2.3 1 / 2

✓ **- 1 pts Missing one**

2.4 0 / 2

✓ **- 2 pts Incorrect**

2.5 2 / 2

✓ **- 0 pts Correct**

QUESTION 3

Rationals 10 pts

3.1 Equivalence relation 7 / 8

✓ **- 1 pts Transitivity proof error: how do you know you didn't divide by 0?**

Should be "ab = ba" for reflexivity.

3.2 Equivalence classes 2 / 2

✓ **- 0 pts Correct**

QUESTION 4

4 Fibonacci numbers 10 / 10

✓ **- 0 pts Correct**

QUESTION 5

Pairs of subsets 10 pts

5.1 Find the function 2 / 2

✓ **- 0 pts Correct**

5.2 Find the pair of subsets 2 / 2

✓ **- 0 pts Correct**

5.3 It is a bijection 5 / 6

✓ **- 1 pts Problem with surjectivity**

need to start with an arbitrary function, how do you determine the subsets?

Midterm 1

Last Name: Wang

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Section:

	Tuesday:	Thursday:	
	1A	1B	TA: Albert Zheng
	1C	1D	TA: Benjamin Spitz
	1E	1F	TA: Eilon Reisin-Tzur

Instructions: Do not open this exam until instructed to do so. You will have 50 minutes to complete the exam. Please print your name and student ID number above, and circle the number of your discussion section. **You may not use calculators**, books, notes, or any other material to help you. Please make sure **your phone is silenced** and stowed where you cannot see it. Remember that you are bound by a conduct code, and that you may not look at anyone's paper or let anyone look at your paper, nor may you work on the exam after it has ended. There is some scratch paper at the back of the exam, if you do work there indicate so on the relevant question. Do not write on the backs of the pages. Please circle or box your final answers.

Please get out your id and be ready to show it when you turn in your exam.

Please do not write below this line.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

1. (10 points) Circle the correct answer (only one answer is correct for each question)

   1. The relation on the integers defined by xRy if $x - y$ is odd is:
      (a) an equivalence relation
      (b) symmetric, but not transitive or reflexive
      (c) reflexive, but not symmetric or transitive
      (d) symmetric and transitive, but not reflexive

      reflex X
      sym ✓
      trans
      $x - y = k$
      $y - z = q$
      $x - z = k + q$

   2. If $f : X \to Y$ is an one-to-one function and $g : Y \to Z$ is an onto function, then $g \circ f$ is necessarily:
      (a) one-to-one
      (b) onto
      (c) a bijection
      (d) not necessarily one-to-one or onto

   3. If $f : X \to Y$ and $g : Y \to Z$ are functions and $g \circ f$ is a bijection, then:
      (a) f must be one-to-one and g must be onto
      (b) f and g must both be bijections
      (c) f must be onto
      (d) g must in one-to-one

      f onto ✓
      x
      y
      z
      g

   4. The function $f : \{a, b\}^* \times \{a, b\}^* \to \{a, b\}^*$ (X^* is the set of strings in X) defined by $f((\alpha, \beta)) = \alpha\beta$ is:
      (a) onto and one-to-one
      (b) one-to-one but not onto
      (c) neither one-to-one nor onto
      (d) onto but not one-to-one

      abba

   5. For X a set with n elements, how many relations are there on X that are both partial orders and equivalence relations?
      (a) more than 1 and less than n
      (b) more than n
      (c) exactly n
      (d) exactly 1

      $X = \{1, 2, 3\}$
      sym & antisym
      $= \{(1,1), (2,2), (3,3)\}$

$g(y) = y$

2. In this question write down your answer, no need for any justification.

(a) (2 points) Give an example of a functions $f : X \to Y$, $g : Y \to Z$ so that g is onto but $g \circ f$ is not onto.

$f(x) = 2x : \mathbb{Z} \to \mathbb{Z}$ g is onto but $g \circ f$ is not

$g(y) = y : \mathbb{Z} \to \mathbb{Z}$

(b) (2 points) What is $\sum_{i=0}^{100} i$? Feel free to write your answer as the product or sum of a few numbers.

$\sum_{i=0}^{n=100} i = \frac{n(n+1)}{2} = \boxed{\frac{100(101)}{2}}$

(c) (2 points) List elements of the equivalence relation on $\{a, b, c, d, e\}$ determined by the partition $\{\{a, b\}, \{c\}, \{d, e\}\}$.

$R = \{ (a,a), (a,b), (b,a), (c,c), (d,d), (d,e), (e,d), (e,e) \}$

(d) (2 points) If X is a set with n elements where $n \geq 1$, how many onto functions are there from X to $\{0, 1\}$?

$\boxed{2^n}$ (each element has 2 options)

(e) (2 points) Give an example of a relation on $X = \{a, b, c\}$ that is reflexive and symmetric but not transitive.

$R = \{ (a,a), (b,b), (c,c), (b,a), (a,b), (b,c), (c,b) \}$

3. Let $X = \mathbb{Z} \times \mathbb{Z}_{>0}$ (i.e. ordered pairs of integers where the second integer is positive), and define a relation Q on X by $(a,b)Q(c,d)$ if $ad = bc$.

(a) (8 points) Show that Q is an equivalence relation on X.

reflexive: if Q is reflexive, $(a,b)Q(a,b)$ needs to be true. By definition,

$ab = ab$ $\therefore$ Q is reflexive

symmetric: if Q is symmetric, then $(a,b)Q(c,d)$ implies that $(c,d)Q(a,b)$.

$(c,d)Q(a,b) \Rightarrow cb = da$ same

$(a,b)Q(c,d) \Rightarrow ad = bc$

$\therefore$ Q is symmetric

transitive: if Q is trans, then if $(a,b)Q(c,d)$ and $(c,d)Q(e,f)$, then $(a,b)Q(e,f)$

$(a,b)Q(c,d) \Rightarrow ad = bc$

$(c,d)Q(e,f) \Rightarrow cf = de$ $c = \frac{de}{f}$

$ad = \frac{bde}{f}$

$\therefore$ Q is transitive

$af = be \Rightarrow (a,b)Q(e,f)$ exists

(b) (2 points) Give three different elements of $[(2,3)]_Q$.

3 elements of $[(2,3)]_Q$ include:

$(2,3)$, $(12,18)$, $(6,9)$

$2d = 3c$

4. (10 points) Recall that the Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$ is defined by $F_0 = 0, F_1 = 1$, and for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$. Show that for $n \geq 0$, $\sum_{i=0}^{n} F_i = F_{n+2} - 1$.

Base case:

$n = 0$

$\sum_{i=0}^{0} F_i = F_{0+2} - 1$ $\quad$ $F_2 = 1 + 0$

$F_0 = F_2 - 1$

$0 = 1 - 1 = 0$ ✓

Inductive step:

Assume that $\sum_{i=0}^{n} F_i = F_{n+2} - 1$ is true.

Now we prove $n+1$ case:

$\sum_{i=0}^{n+1} F_i = F_{n+3} - 1$ $\quad$ $F_{n+3} = F_{n+2} + F_{n+1}$ by definition

$\sum_{i=0}^{n} F_i + F_{n+1} = F_{n+3} - 1$

$F_{n+2} - 1 + F_{n+1} = F_{n+3} - 1$ $\quad$ (inductive hypothesis)

$F_{n+3} - 1 = F_{n+3} - 1$ ✓

$\square$ proved by induction.

5. Let X be a set, and let $P = \{(A, B) \in \mathcal{P}(X) \times \mathcal{P}(X) : A \subseteq B\}$ be the set of ordered pairs (A, B) of subsets of X where $A \subseteq B$. Let $\{0,1,2\}^X$ denote the set of functions from X to $\{0,1,2\}$.

Define a function $F : P \to \{0,1,2\}^X$ by for $(A, B) \in P$ the function $F((A, B)) : X \to \{0,1,2\}$ is defined by for $x \in X$,

$$F((A,B))(x) = \begin{cases} 0 & x \in A \\ 1 & x \in B \text{ and } x \notin A \\ 2 & x \notin B \end{cases}.$$

(a) (2 points) Let $X = \{a, b, c, d\}$. What is $F((\{a,d\}, \{a,b,d\}))$? Remember your answer should be a function $X \to \{0,1,2\}$.

$$F((\{a,d\}, \{a,b,d\}))(x) = \begin{cases} 0 & x \in \{a,d\} \\ 1 & x \in \{a,b,d\} \text{ and } x \notin \{a,d\} \\ 2 & x \notin \{a,b,d\} \end{cases}$$

(b) (2 points) Again let $X = \{a, b, c, d\}$. Find a pair (A, B) of subsets of X with $A \subseteq B$ so that $F((A, B))$ is the function $g : X \to \{0,1,2\}$ defined by $g(a) = 1, g(b) = 1, g(c) = 2, g(d) = 2$.

If $A = \{\} = \emptyset$

$B = \{a, b\}$

$(A,B) = (\{\}, \{a,b\})$

then the conditions are satisfied.

Question 5 continues on the next page...

Question 5 continued...

(c) (6 points) Show that for any set X the function $F : P \to \{0,1,2\}^X$ defined on the previous page is a bijection.

injective:

If F is injective, then if two outputs are the same, then they should have the same input.

$F((A_1,B_1))(x) = F((A_2,B_2))(x)$

$\begin{cases} 0 & x \in A_1 \\ 1 & x \in B_1 \text{ and } x \notin A_1 \\ 2 & x \notin B_1 \end{cases} = \begin{cases} 0 & x \in A_2 \\ 1 & x \in B_2 \text{ and } x \in A_2 \\ 2 & x \notin B_2 \end{cases}$

In this case, A_1 has to equal A_2 and B_1 has to equal B_2 so F is injective.

surjective:

If F is surjective, every function in $g \in \{0,1,2\}^X$ must be mapped to (must be an input for every output)

$g = \begin{cases} 0 & x \in A \\ 1 & x \in B \text{ and } x \notin A \\ 2 & x \notin B \end{cases}$

Any possible function g can be represented by A and B b/c A, B are subsets of X where $A \subseteq B$ (A, B come from the power set)

∴ F is surjective

B/c F is both injective & surjective, it is a bijection

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