Math 61-1 Midterm 1 version b

[redacted]

TOTAL POINTS

39 / 50

QUESTION 1

Multiple choice 10 pts

1.1 **2 / 2**

✓ + **2 pts** Correct (b)

+ **0 pts** Incorrect

1.2 **2 / 2**

✓ + **2 pts** Correct (d)

+ **0 pts** Incorrect

1.3 **2 / 2**

✓ + **2 pts** Correct (a)

+ **0 pts** Incorrect

1.4 **0 / 2**

+ **2 pts** Correct (d)

✓ + **0 pts** Incorrect

1.5 **2 / 2**

✓ + **2 pts** Correct (d)

+ **0 pts** Incorrect

QUESTION 2

Short answer 10 pts

2.1 **1 / 2**

✓ - **1 pts** Other error

Restricting the domain of f to X = {1} would work

2.2 **0 / 2**

+ **2 pts** Correct (5050)

+ **1 pts** 1+2+...+100

+ **1 pts** n(n+1)/2

✓ + **0 pts** Incorrect

2.3 **2 / 2**

✓ - **0 pts** Correct

2.4 **0 / 2**

✓ - **2 pts** Incorrect

2.5 **2 / 2**

✓ - **0 pts** Correct

QUESTION 3

Rationals 10 pts

3.1 Equivalence relation 7 / 8

✓ - **1 pts** Transitivity proof error: how do you know you didn't divide by 0?

Should be "ab = ba" for reflexivity.

3.2 Equivalence classes 1 / 2

✓ - **1 pts** Some elements correct (besides (2,3))

QUESTION 4

4 Fibonacci numbers 10 / 10

✓ - **0 pts** Correct

QUESTION 5

Pairs of subsets 10 pts

5.1 Find the function 1 / 2

✓ - **1 pts** You haven't said where all the elements go

5.2 Find the pair of subsets 2 / 2

✓ - **0 pts** Correct

5.3 It is a bijection 5 / 6

✓ - **1 pts** Problem with surjectivity

you need to start with an arbitrary function

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Midterm 1

Last Name: [redacted]

First Name:

Student ID:

Section:

	Tuesday:	Thursday:	
	1A	1B	TA: Albert Zheng
	1C	1D	TA: Benjamin Spitz
	1E	1F	TA: Eilon Reisin-Tzur

Instructions: Do not open this exam until instructed to do so. You will have 50 minutes to complete the exam. Please print your name and student ID number above, and circle the number of your discussion section. **You may not use calculators,** books, notes, or any other material to help you. Please make sure **your phone is silenced** and stowed where you cannot see it. Remember that you are bound by a conduct code, and that you may not look at anyone's paper or let anyone look at your paper, nor may you work on the exam after it has ended. There is some scratch paper at the back of the exam, if you do work there indicate so on the relevant question. Do not write on the backs of the pages. Please circle or box your final answers.

Please get out your id and be ready to show it when you turn in your exam.

Please do not write below this line.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

1. (10 points) Circle the correct answer (only one answer is correct for each question)

1. The relation on the integers defined by xRy if $x - y$ is odd is:
   (a) an equivalence relation   4 1 1 0
   (b) symmetric, but not transitive or reflexive
   (c) reflexive, but not symmetric or transitive
   (d) symmetric and transitive, but not reflexive

2. If $f : X \to Y$ is an one-to-one function and $g : Y \to Z$ is an onto function, then $g \circ f$ is necessarily:
   (a) one-to-one
   (b) onto
   (c) a bijection
   (d) not necessarily one-to-one or onto

3. If $f : X \to Y$ and $g : Y \to Z$ are functions and $g \circ f$ is a bijection, then:
   (a) f must be one-to-one and g must be onto $g(f(x))$
   (b) f and g must both be bijections
   (c) f must be onto
   (d) g must in one-to-one

4. The function $f : \{a, b\}^* \times \{a, b\}^* \to \{a, b\}^*$ (X^* is the set of strings in X) defined by $f((\alpha, \beta)) = \alpha\beta$ is:
   $\alpha_1\beta_1 = \alpha_2\beta_2$
   (a) onto and one-to-one
   (b) one-to-one but not onto   a ab   aab
   (c) neither one-to-one nor onto
   (d) onto but not one-to-one

5. For X a set with n elements, how many relations are there on X that are both partial orders and equivalence relations?
   (a) more than 1 and less than n symmetric and antisymmetric
   (b) more than n
   (c) exactly n
   (d) exactly 1

2. In this question write down your answer, no need for any justification.

(a) (2 points) Give an example of a functions $f : X \to Y$, $g : Y \to Z$ so that g is onto but $g \circ f$ is not onto.

$X = \{1,2,3\}$ $\quad Y = \{1,2,3\}$ $\quad Z = \{1,2\}$

$f = \{(1,1)\}$ $\quad g = \{(1,1),(2,2),(3,2)\}$

(b) (2 points) What is $\sum_{i=0}^{100} i$? Feel free to write your answer as the product or sum of a few numbers.

$$\sum_{i=0}^{100} i = \frac{100 \cdot 100}{2} = 50 \cdot 100 = \boxed{5000}$$

(c) (2 points) List elements of the equivalence relation on $\{a, b, c, d, e\}$ determined by the partition $\{\{a, b\}, \{c\}, \{d, e\}\}$.

$\{(a,a),(a,b),(b,a),(b,b),(c,c),(d,d),(d,e),(e,d),(e,e)\}$

(d) (2 points) If X is a set with n elements where $n \geq 1$, how many onto functions are there from X to $\{0, 1\}$?

$2^{|X|}$

(e) (2 points) Give an example of a relation on $X = \{a, b, c\}$ that is reflexive and symmetric but not transitive.

$R = \{(a,a),(b,b),(c,c),(b,a),(a,b),(b,c),(c,b)\}$

3. Let $X = \mathbb{Z} \times \mathbb{Z}_{>0}$ (i.e. ordered pairs of integers where the second integer is positive), and define a relation Q on X by $(a, b)Q(c, d)$ if $ad = bc$.

(a) (8 points) Show that Q is an equivalence relation on X.

- For all $(a,b) \in X$, $(a,b)\,Q\,(a,b)$ because $ab = ab$, so Q is reflexive.

- For all $(a,b),(c,d) \in X$, if $(a,b)\,Q\,(c,d)$ then $(c,d)\,Q\,(a,b)$ because if $ad = bc$, then $cb = ad$. This means Q is symmetric.

- For all $(a,b),(c,d),(e,f) \in X$, if $(a,b)\,Q\,(c,d)$ and $(c,d)\,Q\,(e,f)$, then $ad = bc$ and $cf = de$. Multiplying the first equation by the second:

$ad \cdot cf = bc \cdot de$, we get $af = de$ by simplifying, so Q is transitive.

$af = be$

Q is an equivalence relation because it is reflexive symmetric and transitive.

(b) (2 points) Give three different elements of $[(2, 3)]_Q$.

$(2,3)$

$(3,2), (4,6), (10,15)$

6

12

4. (10 points) Recall that the Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$ is defined by $F_0 = 0, F_1 = 1$, and for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$. Show that for $n \geq 0$, $\sum_{i=0}^{n} F_i = F_{n+2} - 1$.

Base Case: $\sum_{i=0}^{0} F_i = F_{n+2} - 1 \Rightarrow 0 = 1 - 1$ which is clearly true.

Inductive Step: Assume that $\sum_{i=0}^{n} F_i = F_{n+2} - 1$. We must show that $\sum_{i=0}^{n+1} F_i = F_{n+3} - 1$.

$$\sum_{i=0}^{n+1} F_i = \sum_{i=0}^{n} + F_{n+1} = F_{n+2} - 1 + F_{n+1}$$

(by the inductive hypothesis)

$$= (F_{n+2} + F_{n+1}) - 1$$

$$= F_{n+3} - 1$$

by def. of fib. sequence and $n \geq 0$.

Thus by the principle of mathematical induction, $\sum_{i=0}^{n} F_i = F_{n+2} - 1$ for all $n \geq 0$. $\square$

5. Let X be a set, and let $P = \{(A, B) \in \mathcal{P}(X) \times \mathcal{P}(X) : A \subseteq B\}$ be the set of ordered pairs (A, B) of subsets of X where $A \subseteq B$. Let $\{0, 1, 2\}^X$ denote the set of functions from X to $\{0, 1, 2\}$.

Define a function $F : P \to \{0, 1, 2\}^X$ by for $(A, B) \in P$ the function $F((A, B)) : X \to \{0, 1, 2\}$ is defined by for $x \in X$,

$$F((A, B))(x) = \begin{cases} 0 & x \in A \\ 1 & x \in B \text{ and } x \notin A \\ 2 & x \notin B \end{cases}.$$

(a) (2 points) Let $X = \{a, b, c, d\}$. What is $F((\{a, d\}, \{a, b, d\}))$? Remember your answer should be a function $X \to \{0, 1, 2\}$.

$F((\{a,d\}, \{a,b,d\})) = \{(a,0),(b,1),(d,0)\}$

(b) (2 points) Again let $X = \{a, b, c, d\}$. Find a pair (A, B) of subsets of X with $A \subseteq B$ so that $F((A, B))$ is the function $g : X \to \{0, 1, 2\}$ defined by $g(a) = 1, g(b) = 1, g(c) = 2, g(d) = 2$.

$A = \phi$

$B = \{a, b\}$

Question 5 continues on the next page...

Question 5 continued...

(c) (6 points) Show that for any set X the function $F : P \to \{0, 1, 2\}^X$ defined on the previous page is a bijection.

If $(A_1, B_1), (A_2, B_2) \in P$, then if $F((A_1, B_1)) = F((A_2, B_2))$, then for all $x \in X$ such that $F((A_1, B_1))(x) = 0$, we know that $x \in A_1$ and if $F((A_1, B_1))(x) = 1$, we know $x \in B_1$ and $x \notin B_1$ and if $F((A_1, B_1)) = 2$, $x \notin B_1$, $x \notin A_1$. This will be the same for (A_2, B_2), so $A_1 = A_2$ and $B_1 = B_2$ so F is injective.

For all $y \in \{0, 1, 2\}^X$, we can get an $(A, B) \in P$ such that $F(A, B) = y$, namely

$$y = \begin{cases} 0 & x \in A \\ 1 & x \in B \text{ and } x \notin A \\ 2 & x \notin B \end{cases}$$

and $A \subseteq B$ because when $y = 0$, x can be in both A and B. Thus, F is surjective

F is a bijection because it is injective and surjective.

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