J Cameron Math 61 Mon, January 28, 2019

Midterm 1

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Section:

	Tuesday:	Thursday:	
	1A	1B	TA: Albert Zheng
	1C	1D	TA: Benjamin Spitz
	1E	1F	TA: Eilon Reisin-Tzur

Instructions: Do not open this exam until instructed to do so. You will have 50 minutes to complete the exam. Please print your name and student ID number above, and circle the number of your discussion section. **You may not use calculators,** books, notes, or any other material to help you. Please make sure **your phone is silenced** and stowed where you cannot see it. Remember that you are bound by a conduct code, and that you may not look at anyone's paper or let anyone look at your paper, nor may you work on the exam after it has ended. There is some scratch paper at the back of the exam, if you do work there indicate so on the relevant question. Do not write on the backs of the pages. Please circle or box your final answers.

Please get out your id and be ready to show it when you turn in your exam.

Please do not write below this line.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

1. (10 points) Circle the correct answer (only one answer is correct for each question)

   1. The relation on the integers defined by xRy if $x - y$ is odd is:
      ~~(a)~~ an equivalence relation
      (b) symmetric, but not transitive or reflexive
      ~~(c)~~ reflexive, but not symmetric or transitive
      (d) symmetric and transitive, but not reflexive

      $x-y$
      $5-4=1$
      $y-z$
      $4-3=1$

   2. If $f : X \to Y$ is an one-to-one function and $g : Y \to Z$ is an onto function, then $g \circ f$ is necessarily:
      (a) one-to-one
      (b) onto
      (c) a bijection
      (d) not necessarily one-to-one or onto

      $g \circ f : X \to Z$
      onto 1-to-1

      f g
      $1 \to a \to x$
      $2 \to b \to y$
      $c \to z$
      $d \to z$

   3. If $f : X \to Y$ and $g : Y \to Z$ are functions and $g \circ f$ is a bijection, then:
      (a) f must be one-to-one and g must be onto
      (b) f and g must both be bijections
      ~~(c)~~ f must be onto
      ~~(d)~~ g must in one-to-one

      $1 \to a \to x$
      $2 \to b \to y$
      $3 \to c \to z$
      $d \to z$

   4. The function $f : \{a, b\}^* \times \{a, b\}^* \to \{a, b\}^*$ (X^* is the set of strings in X) defined by $f((\alpha, \beta)) = \alpha\beta$ is:
      (a) onto and one-to-one
      (b) one-to-one but not onto
      (c) neither one-to-one nor onto
      (d) onto but not one-to-one

      $f(\alpha, \beta) = \alpha\beta$
      $f(\alpha, \lambda) = \alpha$

   5. For X a set with n elements, how many relations are there on X that are both partial orders and equivalence relations?
      (a) more than 1 and less than n
      (b) more than n
      (c) exactly n
      (d) exactly 1

      $X = \{1, 2, 3, \ldots, n\}$
      reflexive symmetric asymmetric transitive
      all must be of the form (x, x)

2. In this question write down your answer, no need for any justification.

(a) (2 points) Give an example of a functions $f : X \to Y$, $g : Y \to Z$ so that g is onto but $g \circ f$ is not onto.

$X = \{1, 2\}$ $Y = \{a, b\}$ $Z = \{x, y\}$

$f = \{(1, a), (2, a)\}$ $g = \{(a, x), (b, y)\}$

(b) (2 points) What is $\sum_{i=0}^{100} i$? Feel free to write your answer as the product or sum of a few numbers.

$\sum_{i=0}^{100} i = 0 + 1 + \cdots + 100 = \frac{100(100+1)}{2} = 50 \cdot 101 = 5050$

(c) (2 points) List elements of the equivalence relation on $\{a, b, c, d, e\}$ determined by the partition $\{\{a, b\}, \{c\}, \{d, e\}\}$.

$R = \{(a,a), (a,b), (b,a), (b,b), (c,c), (d,d), (d,e), (e,d), (e,e)\}$

(d) (2 points) If X is a set with n elements where $n \geq 1$, how many onto functions are there from X to $\{0, 1\}$? $X = \{1, 2, 3, \ldots, n\} \to \{0, 1\}$

$2^n - 2$

(e) (2 points) Give an example of a relation on $X = \{a, b, c\}$ that is reflexive and symmetric but not transitive.

$R = \{(a,a), (b,b), (c,c), (a,b), (b,c), (b,a), (c,b)\}$

3. Let $X = \mathbb{Z} \times \mathbb{Z}_{>0}$ (i.e. ordered pairs of integers where the second integer is positive), and define a relation Q on X by $(a,b)Q(c,d)$ if $ad = bc$.

(a) (8 points) Show that Q is an equivalence relation on X.

Reflexive:

for $(a,b) \in \mathbb{Z}$, $(a,b)\,Q\,(a,b)$ because $ab = ba$. So, Q is reflexive.

Symmetry:

If $(a,b)\,Q\,(c,d)$ then $ad = bc$. But $(c,d)\,Q\,(a,b)$ is also true because $cb = da$ is the same thing as $ad = bc$. So, Q is symmetric.

Transitive:

Let $(a,b)\,Q\,(c,d)$ and $(c,d)\,Q\,(e,f)$. Then $ad = bc$ and $cf = de$. We want to show $(a,b)\,Q(e,f)$, so we have to show $af = be$. Multiplying the 2 equations we already have, we get $(ad)(cf) = (bc)(de) \Rightarrow (af)\cancel{(de)} = (be)\cancel{(cd)} \Rightarrow af = be$. So, $(a,b)\,Q\,(e,f)$ and Q is transitive.

(b) (2 points) Give three different elements of $[(2,3)]_Q$.

$(2,3)$
$(4,6)$
$(6,9)$

4. (10 points) Recall that the Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$ is defined by $F_0 = 0, F_1 = 1$, and for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$. Show that for $n \geq 0$, $\sum_{i=0}^{n} F_i = F_{n+2} - 1$.

Base case:

$n=0$. $\sum_{i=0}^{n} F_i = F_0 = 0$ $\quad F_{n+2} - 1 = F_2 - 1 = 1 - 1 = 0$

So LHS = RHS.

Induction step:

Assume $\sum_{i=0}^{n} F_i = F_{n+2} - 1$

We want to show $\sum_{i=0}^{n+1} F_i = F_{n+3} - 1$

$$\sum_{i=0}^{n+1} F_i = \sum_{i=0}^{n} F_i + F_{n+1} = F_{n+2} - 1 + F_{n+1}$$

$$= (F_{n+1} + F_{n+2}) - 1 = F_{n+3} - 1$$

Since we have proved the induction step, we have proved that $\sum_{i=0}^{n} F_i = F_{n+2} - 1 \quad \forall n \geq 0$.

5. Let X be a set, and let $P = \{(A,B) \in \mathcal{P}(X) \times \mathcal{P}(X) : A \subseteq B\}$ be the set of ordered pairs (A,B) of subsets of X where $A \subseteq B$. Let $\{0,1,2\}^X$ denote the set of functions from X to $\{0,1,2\}$.

Define a function $F : P \to \{0,1,2\}^X$ by for $(A,B) \in P$ the function $F((A,B)) : X \to \{0,1,2\}$ is defined by for $x \in X$,

functions $\{0,1,2\}^X = X \to \{0,1,2\}$

input

$$F((A,B))(x) = \begin{cases} 0 & x \in A \\ 1 & x \in B \text{ and } x \notin A \\ 2 & x \notin B \end{cases}.$$

output is a function

$F: \underset{\substack{\text{ordered pair} \\ (A,B)}}{P} \to \underset{\substack{\text{function} \\ X \to \{0,1,2\}}}{\{0,1,2\}^X}$

(a) (2 points) Let $X = \{a,b,c,d\}$. What is $F((\{a,d\},\{a,b,d\}))$? Remember your answer should be a function $X \to \{0,1,2\}$.

$$F((\{a,d\},\{a,b,d\})) = \begin{cases} 0 & \text{if } x = a,d \\ 1 & \text{if } x = b \\ 2 & \text{if } x = c \end{cases}$$

(b) (2 points) Again let $X = \{a,b,c,d\}$. Find a pair (A,B) of subsets of X with $A \subseteq B$ so that $F((A,B))$ is the function $g : X \to \{0,1,2\}$ defined by $g(a) = 1, g(b) = 1, g(c) = 2, g(d) = 2$.

$A = \phi \quad B = \{a,b\}$

Question 5 continues on the next page...

Question 5 continued...

(c) (6 points) Show that for any set X the function $F : P \to \{0, 1, 2\}^X$ defined on the previous page is a bijection.

Let $(A,B), (C,D) \in P$ and let $F((A,B)) = F((C,D))$. So:

$$F((A,B)) = \begin{cases} 0 & x \in A \\ 1 & x \in B\ x \notin A \\ 2 & x \notin B \end{cases} = F((C,D)) = \begin{cases} 0 & x \in C \\ 1 & x \in D\ x \notin C \\ 2 & x \notin D \end{cases}$$

for the 2 functions to be equal, first the output of the functions must be 0 for the same $x \in X$. so, $A = C$. Next the outputs must be 2 for the same $x \in X$. so, $X \setminus B = X \setminus D$, so $B = D$. we have shown that if $F((A,B)) = F((C,D))$, then (A,B) must equal (C,D), so F is 1-to-1.

let $f(x) \in \{0,1,2\}^X$

we want to show $\forall f(x) \in \{0,1,2\}^X$, there exists an $(A,B) \in P$ s.t. $F((A,B)) = f(x)$.

$f(x)$ is $X \to \{0,1,2\}$. So all possible $f(x)$ is defined by how $x \in X$ is assigned to the values $0,1,2$. So for each distinct $f(x)$, there is a set $L \subseteq X$ where all its values are assigned to 0, $M \subseteq X$ where all its values are assigned to 1, and $N \subseteq X$ where all its values are assigned to 2, where L, M, N are pairwise disjoint and $L \cup M \cup N = X$. This can be uniquely mapped to an (A,B) where $A = L$, $B = L \cup M$, and $(A,B) \in P$ since $A \subseteq B$. So $\forall f(x) \in \{0,1,2\}^X$, there exists a unique $(A,B) \in P$ s.t. $F((A,B)) = f(x)$ so F is onto.

Thus F is both 1-to-1 and onto, so F is a bijection.

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