Midterm 1

Last Name:

First Name:

Student ID:

Section:

Instructions: Do not open this exam until instructed to do so. You will have 50 minutes to complete the exam. Please print your name and student ID number above, and circle the number of your discussion section. **You may not use calculators,** books, notes, or any other material to help you. Please make sure **your phone is silenced** and stowed where you cannot see it. Remember that you are bound by a conduct code, and that you may not look at anyone's paper or let anyone look at your paper, nor may you work on the exam after it has ended. There is some scratch paper at the back of the exam, if you do work there indicate so on the relevant question. Do not write on the backs of the pages. Please circle or box your final answers.
Please get out your id and be ready to show it when you turn in your exam.

Please do not write below this line.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

1. (10 points) Circle the correct answer (only one answer is correct for each question)

   1. If $f : X \to Y$ and $g : Y \to Z$ are functions and $g \circ f$ is a bijection, then:
      one-to-one
      onto
      (a) f must be onto
      (b) f must be one-to-one and g must be onto $X = \{1,2\}$ $f: \{(1,2),(2,3)\}$
      (c) g must in one-to-one $Y = \{2,3\}$
      (d) f and g must both be bijections $Z = \{5\}$ $g: \{(2,5),(3,5)\}$
   2. The relation on the integers defined by xRy if $x - y$ is odd is:
      (a) reflexive, but not symmetric or transitive x   1 4   4 7
      (b) symmetric, but not transitive or reflexive
      (c) symmetric and transitive, but not reflexive
      (d) an equivalence relation X
   3. If $f : X \to Y$ is an one-to-one function and $g : Y \to Z$ is an onto function, then $g \circ f$ is necessarily:
      (a) not necessarily one-to-one or onto   onto
      not one-to-one
      (b) one-to-one
      (c) a bijection
      (d) onto
   4. For X a set with n elements, how many relations are there on X that are both partial orders and equivalence relations?
      ref. trans.
      (a) exactly 1   Symmetric & antisymmetric
      (b) more than n $\{1,2\}$ $\{1\}$
      (c) more than 1 and less than n $\{(1,1)\}$
      (d) exactly n $\{(1,1),(2,2)\}$
   5. The function $f : \{a,b\}^* \times \{a,b\}^* \to \{a,b\}^*$ (X^* is the set of strings in X) defined by $f((\alpha, \beta)) = \alpha\beta$ is:
      (a) onto but not one-to-one
      (b) one-to-one but not onto
      (c) neither one-to-one nor onto
      (d) onto and one-to-one

2. In this question write down your answer, no need for any justification.

(a) (2 points) List elements of the equivalence relation on $\{a, b, c, d, e\}$ determined by the partition $\{\{a, b\}, \{c\}, \{d, e\}\}$.

$\{(a,a),(b,b),(a,b),(b,a),(c,c),(d,d),(d,e),(e,d),(e,e)\}$

(b) (2 points) Give an example of a relation on $X = \{a, b, c\}$ that is reflexive and symmetric but not transitive.

$\{(a,a),(b,b),(c,c),(a,b),(b,a),(b,c),(c,b)\}$

(c) (2 points) Give an example of a functions $f : X \to Y$, $g : Y \to Z$ so that g is onto but $g \circ f$ is not onto.

$X = \{\}$ $\quad f(x) = y \quad x \in X$ and $y \in Y$

$Y = \{1,2,3\}$ $\quad g(y) = z \quad y \in Y$ and $z \in Z$

$Z = \{1,2,3\}$

(d) (2 points) If X is a set with n elements where $n \geq 1$, how many onto functions are there from X to $\{0, 1\}$?

$2^n - 2$

(e) (2 points) What is $\sum_{i=0}^{100} i$? Feel free to write your answer as the product or sum of a few numbers.

$0+1+2+3+\cdots+100 = \frac{(100)(101)}{2}$

3. (10 points) Recall that the Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$ is defined by $F_0 = 0, F_1 = 1$, and for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$. Show that for $n \geq 0$, $\sum_{i=0}^{n} F_i = F_{n+2} - 1$.

Basis step:

$$F_0 = 0, \quad F_1 = 1 \implies F_2 = F_0 + F_1 = 0 + 1 = 1$$

$$\sum_{i=0}^{0} F_i = F_2 - 1 = 1 - 1 = 0 \checkmark$$

Inductive step:

our inductive hypothesis is $\sum_{i=0}^{n} F_i = F_{n+2} - 1$ for some n.

we must prove that $\sum_{i=0}^{n+1} F_i = F_{(n+1)+2} - 1$.

$$\sum_{i=0}^{n+1} F_i = F_{n+2} - 1 + F_{n+1}$$

$$= F_{n+3} - 1 \qquad \text{since } F_{n+3} = F_{n+2} + F_{n+1}$$

$$= F_{(n+1)+2} - 1$$

$\square$ Proved by induction

4. Let $X = \mathbb{Z} \times \mathbb{Z}_{>0}$ (i.e. ordered pairs of integers where the second integer is positive), and define a relation Q on X by $(a, b)Q(c, d)$ if $ad = bc$.

(a) (8 points) Show that Q is an equivalence relation on X.

(1,3)Q(1,3)

(1,3)Q(2,6)

Q is reflexive. If (a,b) ∈ X, then (a,b) Q (a,b) since ab = ab.

Q is also symmetric. Q is defined on X where (a,b),(c,d) ∈ X by (a,b) Q (c,d) if ad = bc, so if (a,b) Q (c,d), ad = bc, and bc = ad, so (c,d) Q (a,b). Thus, (c,d) Q (a,b) is also in the relation.

Q is transitive. If (a,b), (c,d), (e,f) ∈ X where (a,b) Q (c,d) and (c,d) Q (e,f), then ad = bc and cf = de. From algebraic manipulation, $\frac{d}{c} = \frac{b}{a} = \frac{f}{e}$, so af = be, so (a,b) Q (e,f), and Q is transitive.

Because Q is reflexive, symmetric, and transitive, it is an equivalence relation on X.

(b) (2 points) Give three different elements of $[(2, 3)]_Q$.

(2,3) R (x,y)

2y = 3x

(2,3), (4,6), (6,9)

5. Let X be a set, and let $P = \{(A, B) \in \mathcal{P}(X) \times \mathcal{P}(X) : A \subseteq B\}$ be the set of ordered pairs (A, B) of subsets of X where $A \subseteq B$. Let $\{0, 1, 2\}^X$ denote the set of functions from X to $\{0, 1, 2\}$.

Define a function $F : P \to \{0, 1, 2\}^X$ by for $(A, B) \in P$ the function $F((A, B)) : X \to \{0, 1, 2\}$ is defined by for $x \in X$,

$$F((A,B))(x) = \begin{cases} 0 & x \in A \\ 1 & x \in B \text{ and } x \notin A \\ 2 & x \notin B \end{cases}.$$

(a) (2 points) Let $X = \{a, b, c, d\}$. What is $F((\{a, d\}, \{a, b, d\}))$? Remember your answer should be a function $X \to \{0, 1, 2\}$.

$F((\{a,d\}, \{a,b,d\})) = \{(a,0), (b,1), (c,2), (d,0)\}$

$F(a) = 0$
$F(b) = 1$
$F(c) = 2$
$F(d) = 0$

(b) (2 points) Again let $X = \{a, b, c, d\}$. Find a pair (A, B) of subsets of X with $A \subseteq B$ so that $F((A, B))$ is the function $g : X \to \{0, 1, 2\}$ defined by $g(a) = 1, g(b) = 1, g(c) = 2, g(d) = 2$.

$A = \{\}$
$B = \{a, b\}$

Question 5 continues on the next page...

Question 5 continued...

(c) (6 points) Show that for any set X the function $F : P \to \{0, 1, 2\}^X$ defined on the previous page is a bijection.

F is one-to-one because for every function f from X to $\{0,1,2\} \in \{0,1,2\}^X$ there exists a single ~~[illegible]~~ $(A,B) \in P$ such that

$$F((a,b))(x) = \begin{cases} 0 & x \in A \\ 1 & x \in B \text{ and } x \notin A \\ 2 & x \notin B \end{cases}$$

F is onto because ~~there exis~~ for every function $X \to \{0,1,2\}$ there exists an ~~[illegible]~~ $(A,B) \in P$ such that $P \to \{0,1,2\}^X$.

Because F is both one-to-one and onto, F is a bijection.

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