Midterm 1

Last Name: ____________________

First Name: ____________________

Student ID: ____________________

Section:

	Tuesday:	Thursday:	
	2A	2B	TA: Alex Mennen
	2C	2D	TA: Van Latimer

Instructions: Do not open this exam until instructed to do so. You will have 50 minutes to complete the exam. Please print your name and student ID number above, and circle the number of your discussion section. **You may not use calculators,** books, notes, or any other material to help you. Please make sure **your phone is silenced** and stowed where you cannot see it. Remember that you are bound by a conduct code, and that you may not look at anyone's paper or let anyone look at your paper. You may use any available space on the exam for scratch work. If you need more scratch paper, please ask one of the proctors. There is some scratch paper at the back of the exam. Please circle or box your final answers. **Please get out your id and be ready to show it when you turn in your exam.**

Please do not write below this line.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

1. (10 points) 1. If X has 5 elements and Y has 3 elements, the number of relations from X to Y is:
   (a) $2^5 3$
   (b) $2^5 2^3$
   (c) $2^{5\times 3}$– **Answer**
   (d) $2^5 + 2^3$

2. If $X = \{1, 2, 3, 4\}$ then the relation $R = \{(2, 3), (3, 2), (2, 2)\}$ is:
   (a) Antisymmetric
   (b) Reflexive
   (c) Symmetric– **Answer**
   (d) Transitive

3. If R is a partial order on a set X, then R^{-1} is necessarily:
   (a) A partial order– **Answer**
   (b) Symmetric
   (c) An equivalence relation
   (d) A function

4. The function $f : \{a, b\}^* \to \{a, b\}^*$ (X^* is the set of strings in X) defined by $f(\alpha) = \alpha\alpha$ is:
   (a) onto and one-to-one
   (b) one-to-one but not onto– **Answer**
   (c) neither one-to-one nor onto
   (d) onto but not one-to-one

5. $\frac{10!}{8!}$ is
   (a) more than 200
   (b) more than 100 but less than 200
   (c) more than 50 but less than 100– **Answer**
   (d) more that 10 but less than 50

2. In this question write down your answer, no need for any justification. Please leave answers in the form of factorials, $C(n,m)$ or $\binom{n}{m}$, $P(n,m)$, and with exponents instead of multiplying the answer out.

   (a) (2 points) What is the number of ways of rearranging the letters of COMBINATORICS?

   $\binom{13}{2}\binom{11}{2}\binom{9}{2}7! = \frac{13!}{2!2!2!}$

   (b) (2 points) What is the number of ways to arrange eight math majors and five CS majors in a line, if the CS majors will not stand next to each other?

   $8!P(9,5)$

   (c) (2 points) What is the number of way to distribute 10 identical acorns between 4 different squirrels?

   $\binom{13}{3}$

   (d) (2 points) What is the number of one-to-one functions from $\{1,2,3\}$ to $\{a,b,c,d,e\}$ such that a and b are both in the range of the function?

   $3 \cdot 2 \cdot 3$

   (e) (2 points) What is the coefficient of x^3y^7 in $(2x-3y)^{10}$?

   $\binom{10}{3}2^3(-3)^7$

3. (10 points) Use induction to show that $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$. Be sure to show your work and justify your answer.

To check the base case, we verify that $1^3 = \frac{1(2)}{2}$.

Now for the inductive step, suppose that $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$, and consider $\sum_{i=1}^{n+1} i^3$. This is $\sum_{i=1}^{n} i^3 + (n+1)^3$, so by our inductive hypothesis

$$\sum_{i=1}^{n+1} i^3 = \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3.$$

Putting the right hand side over a common denominator and factoring out an $(n+1)^2$ yields:

$$\frac{(n+1)^2(n^2+4(n+1))}{2^2}.$$

But since $n^2 + 4n + 4 = (n+2)^2$, putting this all together we have:

$$\sum_{i=1}^{n+1} i^3 = \left(\frac{(n+1)(n+2)}{2}\right)^2.$$

This is what we wanted to show, so we are done by induction.

4. Consider the relation on the real numbers defined by xRy if $x - y$ is an integer.

   (a) (6 points) Show that R is an equivalence relation.

   For any real number x, $x - x = 0$ which is an integer, so R is reflexive.

   If xRy then $x - y$ is an integer, so $-(x - y) = y - x$ is an integer as well, so we have that yRx, so R is symmetric.

   If xRy and yRz, then $x - y$ and $y - z$ are both integers. But $x - z = (x - y) + (y - z)$. The sum of two integers is an integer, so $x - z$ is an integers and xRz, so R is transitive.

   Therefore, R is an equivalence relation.

   (b) (4 points) Write down 3 distinct elements in the equivalence class of $1/4$.

   $1/4, 100.25, -3/4$

5. (a) (5 points) Show that if $f : X \to Y$ and $g : Y \to Z$ are functions and f and g are both onto, then $g \circ f$ is onto. Justify your work!

Suppose that f and g are onto, and consider a $z \in Z$. Since g is ont, there exists a $y \in Y$ with $g(y) = z$. Since f is onto, there is an $x \in X$ so that $f(x) = y$. Therefore $g \circ f(x) = z$, so $g \circ f$ is onto.

(b) (5 points) Show that $\sum_{i=0}^{n} 2^i \binom{n}{i} = 3^n$ (remember $\binom{n}{i}$ means the same thing as $C(n, i)$). Justify your work!

By the binomial theorem $(1 + 2)^n = \sum_{i=0}^{n} \binom{n}{i} 2^i 1^{n-i}$. The left hand side is 3^n and the right hand side is $\sum_{i=0}^{n} \binom{n}{i} 2^i$, so we are done.