Midterm 1 2

Last Name: [redacted]

First Name:

Student ID:

Section:

	Tuesday:	Thursday:	
	2A	2B	TA: Alex Mennen
	2C	2D	TA: Van Latimer

Instructions: Do not open this exam until instructed to do so. You will have 50 minutes to complete the exam. Please print your name and student ID number above, and circle the number of your discussion section. **You may not use calculators,** books, notes, or any other material to help you. Please make sure **your phone is silenced** and stowed where you cannot see it.
Remember that you are bound by a conduct code, and that you may not look at anyone's paper or let anyone look at your paper. If you write on the exam before the exam starts or after it end, this will be considered and act of academic dishonesty.
You may use any available space on the exam for scratch work. If you need more scratch paper, please ask one of the proctors. Please circle or box your final answers.
Please get out your id and be ready to show it when you turn in your exam.

Please do not write below this line.

Question	Points	Score
1	10	8
2	10	10
3	10	8
4	10	10
5	10	9
Total:	50	45

1. (10 points) The follow questions have one correct answer, indicate which answer is correct.

   1. If X and Y are finite sets and every function from X to Y is not injective, then:

      not one-to-one

      (a) $|X| < |Y|$

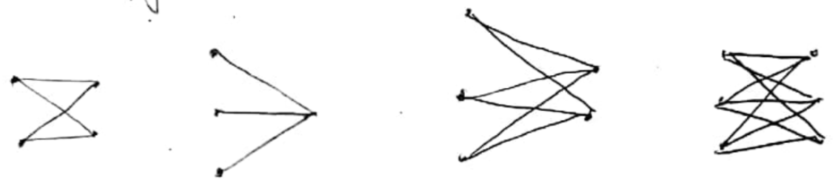
      (b) $|X| > |Y|$ ← Pigeonhole

      (c) $|X| = |Y|$

      (d) $|X|$ could be any of the following: larger than, smaller than, or equal to $|Y|$

      if $|X| = |Y|$, must have SOME injective function (same # distinct elems)

      injective → one-to-one



x

$x = f_{100}\sqrt{5} + 2\left(\frac{1-\sqrt{5}}{2}\right)^{100}$

   2. The complete bipartite graph $K_{m,n}$ has an Euler cycle when:

      $x - \left(\frac{1-\sqrt{5}}{2}\right)^{100} = f_{100}\sqrt{5} + \left(\frac{1-\sqrt{5}}{2}\right)^{100}$

      (a) m and n are both odd

      (b) m and n are both even

      (c) One of m and n is odd and the other is even

      (d) One of m and n is two

      need even degrees, connected

      must be connected

      m, n

      m vertices, each degree n

      n vertices, each degree m

      m is even – n vertices even degree

      n is even – m vertices even degree

      $\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)^{100} + \left(\frac{1-\sqrt{5}}{2}\right)^{100}$

      $f_1 = 1$

      $f_2 = 1$

      $f_n = f_{n-1} + f_{n-2}$

      $f_n - f_{n-1} - f_{n-2} = 0 \quad n \geq 3$

   3. Which of the following is an integer?

      (a) $\left(\frac{1+\sqrt{5}}{2}\right)^{100} - \left(\frac{1-\sqrt{5}}{2}\right)^{100}$ $= f_{100}\cdot\sqrt{5}$

      Solve $t^2 - t - 1 = 0$ $\quad \frac{1 \pm \sqrt{1-4(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$

      $U_n = a\left(\frac{1+\sqrt{5}}{2}\right)^n + b\left(\frac{1-\sqrt{5}}{2}\right)^n$

      $a\left(\frac{2+2\sqrt{5}}{4}\right) + b\left(\frac{2-2\sqrt{5}}{4}\right)$

      (b) $\left(\frac{1+\sqrt{5}}{2}\right)^{100} + \left(\frac{1-\sqrt{5}}{2}\right)^{100}$

      $(a+b)^n + (a-b)^n$

      $a = \frac{1}{2} \quad b = \frac{\sqrt{5}}{2}$ Must have

      $= a\left(\frac{b+2\sqrt{5}}{4}\right) + b\left(\frac{b-2\sqrt{5}}{4}\right)$

      (c) $\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{100} + \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{100}$

      int

      $U_1 = a\left(\frac{1+\sqrt{5}}{2}\right) + b\left(\frac{1-\sqrt{5}}{2}\right) = 1$

      (d) $\left(\frac{1+\sqrt{5}}{2}\right)^{100}$

      $(a+b)^{100} = \sum_{k=0}^{100} C\binom{100}{k} a^{100-k} b^k$

      $= U_2 = a\left(\frac{1+2\sqrt{5}+5}{4}\right) + b\left(\frac{1-2\sqrt{5}+5}{4}\right)$

      $0 = a + b$

      $a = -b$

      $2 - 6 - 2\sqrt{5} + 2\sqrt{5}$

      $a\sqrt{5} = 1$ $\quad a = \frac{1}{\sqrt{5}} \quad b = -\frac{1}{\sqrt{5}}$

      $\frac{1+\sqrt{5}}{2} + \frac{\sqrt{5}-1}{2} = \frac{2\sqrt{5}}{2} = \sqrt{5}$

      $a\left(\frac{1+1}{2}\right) = 1$ $\quad a = 1 \quad b = -1$

Question 1 continues on the next page... $\frac{2}{2} = 1 = U_1$

$\longrightarrow f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$ $\quad \frac{12}{4} = 3 = U_2$

4. Consider the sequence with first term a_0 defined for $n \geq 3$ by the recurrence relation $\boxed{a_n = a_{n-1} + 2a_{n-3}.}$ If $a_1 = 1$, $a_2 = 4$, and $a_3 = 2$, what is a_0?
   (a) $a_0 = 1$
   (b) $a_0 = 0$
   (c) $a_0 = -1$
   (d) Not enough information is given to determine the answer.

What is a_0?
$a_3 = a_2 + 2a_0$
$2 = 4 + 2a_0 \quad -2 = 2a_0 \quad a_0 = -1$

$$a_3 = a_2 + 2a_0$$
$$\parallel$$
$$2 = 4 + 2(a_0) \qquad 2(a_0) = 2 - 4 = -2$$
$$a_0 = -1$$

5. What is the sum of the degrees of the vertices in K_n?
   (a) $n(n+1)$
   (b) $(n-1)n$
   (c) $\frac{(n-1)n}{2}$
   (d) $\frac{n(n+1)}{2}$

K_n has $(n-1)+(n-2)+\ldots+2+1$ edges. $= \frac{n(n-1)}{2}$

Sum degrees $= 2[\#\text{ edges}]$

$10 = \frac{5(4)}{2}$

20 deg.

$2\left(\frac{n(n-1)}{2}\right)$

Complete graph K_n

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2. (10 points) Find a formula for the the recurrence relation $a_n = -2a_{n-1} + 8a_{n-2}$ with initial conditions $a_0 = 2$, $a_1 = -2$.

lhrrcc.

$a_n = -2a_{n-1} + 8a_{n-2}$

test solution of form $t^n = a_n$, find t.

$t^n = -2t^{n-1} + 8t^{n-2}$

1 -2
1 4

$t^{n-2}(t^2 + 2t - 8) = 0$

Solve $t^2 + 2t - 8 = 0$ ✓

$= (t-2)(t+4) = 0$

Lin comb. of solutions.

$t = 2, -4$ ✓

$U_n = a2^n + b(-4)^n$ ✓, $a, b \in \mathbb{R}$ constants

Use initial conditions:

$U_0 = a + b = 2$ ✓

$U_1 = a(2) - 4b = -2$ ✓

$$\begin{array}{r} 2a + 2b = 4 \\ - \; 2a - 4b = -2 \\ \hline 6b = 6 \end{array}$$

$b = 1, a = 1$

Indeed $1 + 1 = 2$, $2 - 4 = -2$

Explicit formula is thus

$U_n = a_n = (1)2^n + (1)(-4)^n = \boxed{2^n + (-4)^n}$ ✓

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3. An *Euler path* in a graph is a path (not necessarily a cycle) that visits each edge in the graph exactly one time.

(a) (7 points) Show that a connected graph G has an Euler path if and only if either every vertex in G has even degree, or G has exactly two vertices of odd degree.

Case 1: IF connected graph G has every vertex with even degree, then G has an Euler cycle. An Euler cycle visits each edge in G exactly once by definition, so it is also an Euler path → thus G has Euler path.

Case 2: IF connected graph G has exactly two vertices of odd degree, call them v_i and v_j, then an Euler cycle can be constructed from it by adding an edge incident on v_i and v_j, since this leaves the graph connected (not removing any existing edges) but makes v_i and v_j have even degree, so that ALL vertices in modified G have even degree → Euler cycle created. This means there is Euler path from v_i to v_j in the absence of the added edge, i.e. in G.

(b) (3 points) Find an Euler path in the following graph. Decribe your path by listing the vertices in the order that they appear in the path.

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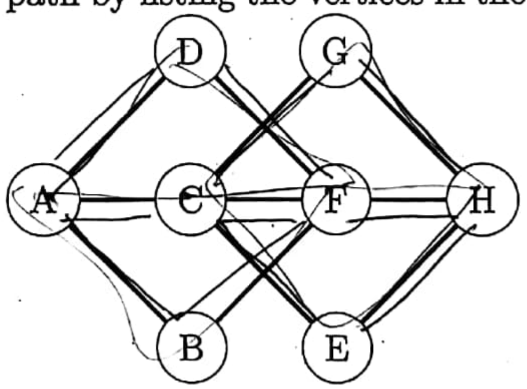

(A, D, F, B, A, C, F, H, E, C, G, H) 3

since there is Euler path from v_i to v_j including extra edge, there is Euler path v_i to v_j after removing extra edge.

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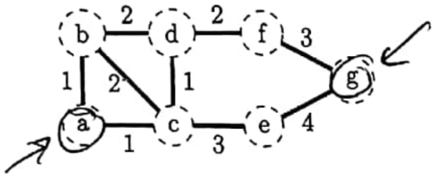
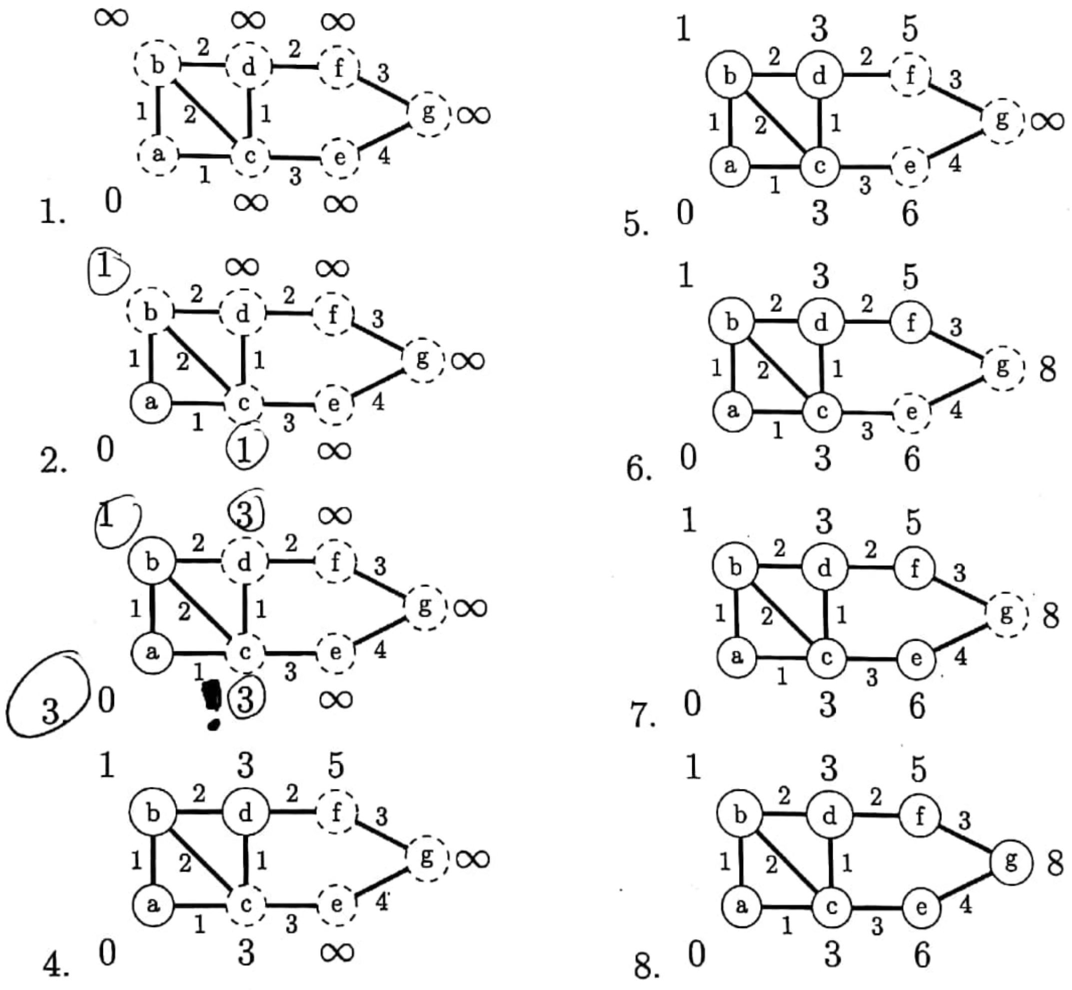
4. (10 points) I am using Dijkstra's algorithm to find the length of the shortest path from vertex a to vertex g in the following weighted graph. When I circle a vertex, I fill in the dotted circles around each vertex.



At what stage do I make an error, and what is my error?



Stage 3: overwrote value of $L(c)$ from 1 to 3 even though shortest path length to c of 1 was already determined in stage 2, and $1 < 3$. (Should only replace if alternate route gives shorter path length.)

what are you using pigeonhole on?

at most $\frac{n(n-1)}{2}$ edges

$\to n(n-1)$ degrees n^2-n

5. (a) (4 points) Show that in any simple graph with two or more vertices, there must be at least two vertices that have the same degree. max degree: $n-1$

Alternative

n vertices

$d_1, d_2, \ldots d_n$ have values $0 \to n-1$ (n possibilities)

$n(n-1)$ ~~degrees max~~

$n^2 > n^2-n$ so by Pigeonhole, at least two vertices have same degree

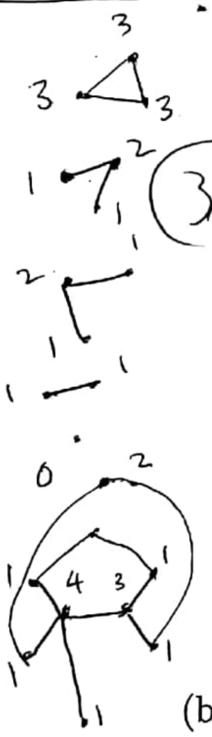
$(n \geq 2)$

Simple graph: no loops, parallel edges

Induct on # vertices, i. Prove for $i \geq 2$

Base case: $i = 2$ Two possibilities: Both degree 0 ✓

Both degree 1 ✓



3/4

Assume graph w/ $i-1$ vertices has at least 2 vertices w same degree.

Add 1 vertex to get a graph with i vertices.

Since graph is simple, the added vertex may or may not make single edges with each of the $i-1$ other vertices.

If it makes edges only with vertices of distinct degree, since the $i-1$ subgraph has ≥ 2 vertices same degree, i graph also does.

If it makes edges with [illegible] both of the vertices with same degree, both degrees increase by 1 and are still equal. If it makes edge with only one of vertices with same degree,

n^2-n degrees max

n vertices, each has n possible ~~degree values~~

n^2 possible comb. of the vertices

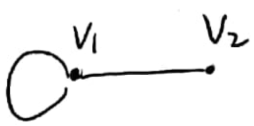
(b) (2 points) Is this true of all graphs, not just simple ones? Either prove that it is true of all graphs or give a counterexample.

since total sum of degrees must be even

2/2

No.

Consider this graph with 2 vertices (has loop, therefore not simple)



V_1 has degree $2+1=3$

V_2 has degree 1 $\quad 1 \neq 3$

$\to$ no two vertices have the same degree. This is a counterexample.

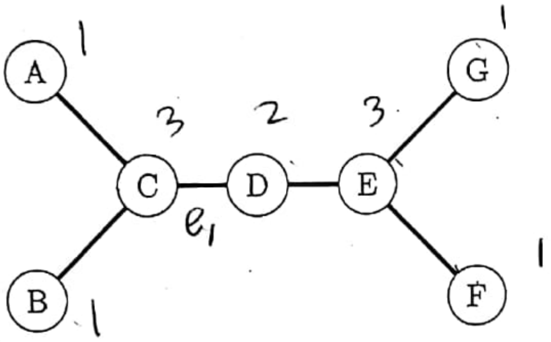
n^2 max total degrees...? No...

Cannot have both vertex degree 0 and $n-1$

$\to$ only $n-1$ possible degrees for n vertices

Question 5 continues on the next page...

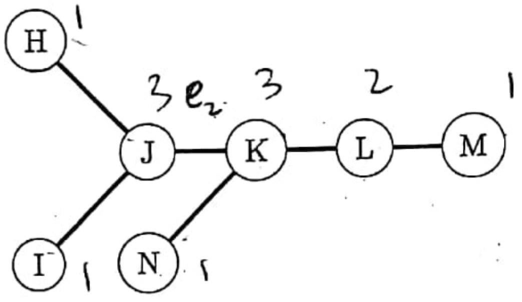
Question 5 continued...



Graph One

7 vertices

6 edges



Graph Two

(c) (4 points) Are the above two graphs isomorphic? Be sure to justify your answer.

4/4

Not isomorphic.

Graph 2 has an edge incident on two vertices ~~degrees~~ with degree 3, J & K (labelled e_2 in diagram).

If Graph 1 were isomorphic to Graph 2, it would also have such an edge (invariant property), but it does not. (By inspection, no edges in Graph 1 ~~[illegible]~~ are incident on two vertices, each of degree 3.)

Non-equal adjacency matrices

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