J Cameron | Math 61 | Mon, Oct 22, 2018

Midterm 1

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Section:

	Tuesday:	Thursday:	
	2A	2B	TA: Alex Mennen
	2C	2D	TA: Van Latimer

Instructions: Do not open this exam until instructed to do so. You will have 50 minutes to complete the exam. Please print your name and student ID number above, and circle the number of your discussion section. **You may not use calculators,** books, notes, or any other material to help you. Please make sure **your phone is silenced** and stowed where you cannot see it. Remember that you are bound by a conduct code, and that you may not look at anyone's paper or let anyone look at your paper. You may use any available space on the exam for scratch work. If you need more scratch paper, please ask one of the proctors. There is some scratch paper at the back of the exam. Please circle or box your final answers. **Please get out your id and be ready to show it when you turn in your exam.**

Please do not write below this line.

Question	Points	Score
1	10	10
2	10	10
3	10	10
4	10	10
5	10	9
Total:	50	49

1. (10 points) 1. If X has 5 elements and Y has 3 elements, the number of relations from X to Y is:

   (a) $2^5 3$

   (b) $2^5 2^3$

   (c) $2^{5\times 3}$

   (d) $2^5 + 2^3$

   X: 1 2 3 4 5   Y: 1 2 3   possible $5 \cdot 3$

2. If $X = \{1, 2, 3, 4\}$ then the relation $R = \{(2, 3), (3, 2), (2, 2)\}$ is:

   (a) Antisymmetric

   (b) Reflexive

   (c) Symmetric

   (d) Transitive

   $(3,2)\ (2,3) \Rightarrow \cancel{(3,3)}$

3. If R is a partial order on a set X, then R^{-1} is necessarily:

   (a) A partial order

   (b) Symmetric

   (c) An equivalence relation

   (d) A function

   reflexive, transitive, antisymm. → $\{1,2\}\ \{1,1\}\ \{2,2\}$ $\{(2,1)\ (1,1)\ (2,2)\}$ not symm. not funct.

4. The function $f : \{a, b\}^* \to \{a, b\}^*$ (X^* is the set of strings in X) defined by $f(\alpha) = \alpha\alpha$ is:

   (a) onto and one-to-one

   (b) one-to-one but not onto

   (c) neither one-to-one nor onto

   (d) onto but not one-to-one

   $f(a)$: a → aa, b → bb, ab abab. can't make a, b

5. $\dfrac{10!}{8!}$ is

   (a) more than 200

   (b) more than 100 but less than 200

   (c) more than 50 but less than 100

   (d) more that 10 but less than 50

   $10 \cdot 9 = 90$

2. In this question write down your answer, no need for any justification. Please leave answers in the form of factorials, $C(n,m)$ or $\binom{n}{m}$, $P(n,m)$, and with exponents instead of multiplying the answer out.

(a) (2 points) What is the number of ways of rearranging the letters of COMBINATORICS?

2 C's
2 O's
1 M
1 B
2 I's
1 N
1 A
1 T
1 R
1 S

$$\boxed{\frac{13!}{2!\,2!\,2!}}$$

✓

(b) (2 points) What is the number of ways to arrange eight math majors and five CS majors in a line, if the CS majors will not stand next to each other?

8 7 6 5 4 3 2 1

$$\boxed{8! \cdot P(9,5)}$$

✓

9 spots to put 5 CS students
order matters
P(9, 5)

(c) (2 points) What is the number of way to distribute 10 identical acorns between 4 different squirrels?

$$\binom{10+4-1}{4-1} = \boxed{\binom{13}{3}}$$

✓

(d) (2 points) What is the number of one-to-one functions from $\{1,2,3\}$ to $\{a,b,c,d,e\}$ such that a and b are both in the range of the function?

1 a
2 b
3 c
d
e

$\underset{a}{3}$ $\underset{b}{2}$ $\underset{\text{leftover (c, d or e)}}{3}$

$$\boxed{3 \cdot 2 \cdot 3}$$

✓

(e) (2 points) What is the coefficient of x^3y^7 in $(2x-3y)^{10}$?

$$(2x-3y)^{10} \Rightarrow \binom{10}{3}(2x)^3(-3y)^7$$

$$= \binom{10}{3} \cdot 2^3 x^3 \cdot (-3)^7 y^7$$

$$= -2^3 \cdot 3^7 \cdot \binom{10}{3} x^3 y^7 \rightarrow \boxed{-2^3 \cdot 3^7 \cdot \binom{10}{3}}$$

✓

3. (10 points) Use induction to show that $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$. Be sure to show your work and justify your answer.

base case:

$n=1$: $\sum_{i=1}^{1} i^3 = 1^3 = 1$

$\left(\frac{1(1+1)}{2}\right)^2 = \left(\frac{2}{2}\right)^2 = 1$

Now, suppose that $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$. inductive hypothesis.

If we can prove that $\sum_{i=1}^{n+1} i^3 = \left(\frac{(n+1)((n+1)+1)}{2}\right)^2$, then the statement itself is proven through induction.

$$\sum_{i=1}^{n+1} i^3 = \underbrace{\sum_{i=1}^{n} i^3} + (n+1)^3$$

by inductive hypothesis

$$= \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3$$

$$= \frac{n^2(n+1)^2}{4} + \frac{4(n+1)^3}{4}$$

$$= \frac{n^2(n+1)^2 + 4(n+1)(n+1)^2}{4}$$

$$= \frac{(n^2+4(n+1))(n+1)^2}{4}$$

$$= \frac{(n^2+4n+4)(n+1)^2}{4}$$

(more on back)

$$= \frac{(n+2)^2 (n+1)^2}{4}$$

$$= \left(\frac{(n+2)(n+1)}{2}\right)^2$$

✓ $$= \left(\frac{(n+1)((n+1)+1)}{2}\right)^2$$

$\square$.

Since we have shown that there is an established base case and that the statement is true for every time n increases by 1, we have proven by induction that the statement is true for $n \geq 1$, which are the only values n is allowed to be (by virtue of the summation notation). Therefore, the statement is true.

4. Consider the relation on the real numbers defined by xRy if $x - y$ is an integer.

(a) (6 points) Show that R is an equivalence relation.

reflexive?

$x - x = 0$, 0 is an integer.

Therefore, xRx always, so R is reflexive.

Symmetric?

$xRy \Rightarrow x - y = k$, where k is an integer.

(the negative of an integer is always an integer also) $\longrightarrow$ $y - x = -k$, and $-k$ is also an integer $\Rightarrow yRx$,

Therefore, sing xRy implies yRx, it is symmetric

transitive?

$xRy \Rightarrow x - y = k_1$, k_1 is an integer

$yRz \Rightarrow y - z = k_2$, k_2 is an integer

$+$

$\Rightarrow x - z = k_1 + k_2$.

Since the sum of 2 integers is always an integer, then xRz. Therefore, since xRy and yRz implies xRz, it is transitive (more on back)

(b) (4 points) Write down 3 distinct elements in the equivalence class of 1/4.

$[\frac{1}{4}] = \{a : aR\frac{1}{4}\}$

Therefore, $a - \frac{1}{4} = k$ where k must be an integer.

Then, choosing $k = 1, 2, 3$, we get the respective values for a:

$a = \frac{5}{4}, \frac{9}{4}, \frac{13}{4}$

which are distinct elements of the equiv. class for 1/4.

$a - \frac{1}{4} = 1$

$a = \frac{5}{4}$

$a - \frac{1}{4} = 2$

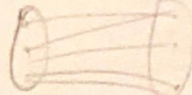
$a = \frac{9}{4}$

$a - \frac{1}{4} = 3$

$a = \frac{13}{4}$

4a. (continued)

Since the definition of an equivalence relation is that it has to be reflexive, symmetric, and transitive and we have shown that all 3 are true in this case, the relation is an equivalence relation.



5. (a) (5 points) Show that if $f: X \to Y$ and $g: Y \to Z$ are functions and f and g are both onto, then $g \circ f$ is onto. Justify your work!

Since g is onto, then that means for every $z \in Z$, there exists at least one $y \in Y$ that fulfills g. Then, since f is onto, the same logic can be applied so that for every $y \in Y$, there exists at least one $x \in X$.

be more precise

From working backwards like this, we can see that there will always be a ? path from any $z \in Z$ to at least one $x \in X$, meaning that the range of the composition of functions g and f, $(g \circ f)$ will always cover Z. Therefore, $g \circ f$ is onto. why?

4/5

(b) (5 points) Show that $\sum_{i=0}^{n} 2^i \binom{n}{i} = 3^n$ (remember $\binom{n}{i}$ means the same thing as $C(n,i)$). Justify your work!

Binomial Thm: $(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$

Looking at the binomial thm, we can see that the structure of the statement and the theorem are very similar. Upon closer inspection, we find that we can obtain the statement by plugging in $a=1$ and $b=2$ into the thm:

5/5

$$(1+2)^n = \sum_{i=0}^{n} \binom{n}{i} (1)^{n-i} (2)^i$$

$$(3)^n = \sum_{i=0}^{n} (2)^i \binom{n}{i}$$

□.