Final Exam

Instructions: Please do each question on a separate page, and make sure that you write neatly and clearly so that it shows up on Gradescope. Remember that you may not discuss the exam with other students, or post the exam questions online in any fashion. If you have questions, please submit a question to the instructors on Piazza or email Professor Cameron.

The exam is due by 9 am Los Angeles time on Sunday 3/14. Double check the time if you are not in LA– daylight savings time starts Sunday morning. Make sure that you also do the multiple choice questions on Gradescope!

1. In this question, write down your answer, no need for any justification. You can leave your answer in terms of factorials, combination symbols, permutation symbols, etc. Please clearly box your answers in your submission to Gradescope.

   (a) How many nonisomorphic (free) trees are there with 4 vertices?

   (b) How many solutions are there to $x_1+x_2+x_3+x_4 = 20$ where $x_4 \geq 10$ and each x_i is a natural number (0 counts as a natural number for us).

   (c) Let $X = \{1, 2, 3, 4, 5\}$. How many relations are there on X with the property that for all $x \in X$, x is not related to itself?

   (d) Give the equivalence relation on $\{a, b, c, d, e\}$ whose equivalence classes give the partition $\mathcal{P} = \{\{a, b\}, \{c, d\}, \{e\}\}$.

   (e) You are dividing 112 apples among 5 boxes. What is the smallest number of apples that could appear in the box with the most apples?

2. Let X be a set with n elements. Be sure to justify all your answers.

   (a) (3 points) How many reflexive relations are there on X?

   (b) (4 points) How many antisymmetric relatoins are there on X?

   (c) (3 points) How many reflexive or antisymmetric relations are there on X?

3. (a) (4 points) Let B_n denote the number of partitions of a set with n elements and set $B_0 = 1$.

   Show that B_n satisfies the recurrence $B_n = \sum_{k=0}^{n-1} \binom{n-1}{k} B_{n-1-k}$.

   (b) (3 points) Show that for all $n \geq 1$, $2^{n-1} \leq B_n$.

   (c) (3 points) Show that for $n \geq 1$, $B_n \leq 2^{n^2}$. **Hint:** Remember that for a nonempty set there is a bijection between partitions on that set and equivalence relations on that set.

4. (a) (5 points) Give a formula for the number of subgraphs of K_n that have exactly n vertices (and prove that it is correct)

   (b) (5 points) Give a formula for the number of subgraphs of K_n (and prove that it is correct).

5. (a) (2 points) Show that every cycle in the n-cube is of length 4 or longer.
   (b) (2 points) How many edges does the n-cube have?
   (c) (2 points) If the n-cube was planar, how many faces would it have? (Your answer will depend on n).
   (d) (4 points) Show that the 4-cube (and hence every n-cube with $n \geq 4$) is not planar.

6. (a) (5 points) Show that $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

   (b) (5 points) Show that if $f : X \to Y$ is onto then there is a function $g : Y \to X$ such that $f \circ g = id_Y$.

7. (a) (5 points) Suppose that G is a simple connected graph with finitely many vertices, and suppose that e is an edge in G such that removing e from G results in a disconnected graph. Show that e is in every spanning tree of G.

   (b) (5 points) Suppose that G is a simple connected weighted graph with finitely many vertices, and that if distinct edges e and e' in G have the same weight, then removing either e or e' from G results in a disconnected graph.

   Show that G has a unique minimal spanning tree.

8. A perfect binary tree of height h is a binary tree of height h with 2^h terminal vertices.

   (a) (5 points) Show that if T is a perfect binary tree of height h then the left and right subtrees of the root are each perfect binary trees of height $h - 1$.

   (b) (5 points) Show that if T_1 and T_2 are each perfect binary trees of height h then T_1 and T_2 are isomorphic as binary trees.

9. A 3-ary tree is a rooted tree where each parent has at most three children, and each child is labeled with 1, 2, or 3 (and siblings all have different labels). A full 3-ary tree is a 3-ary tree where each parent has exactly 3 children.

   Two 3-ary trees are isomorphic as 3-ary trees if they are isomorphic as rooted trees and the isomorphism preserves the labels of the children.

   (a) (5 points) Show that there is a a bijection between the set of nonisomorphic (as 3-ary trees) 3-ary trees with n vertices and the set of nonisomorphic (as 3-ary trees) full 3-ary trees with $2n+1$ terminal vertices.

   (b) (5 points) Let t_n be the number of nonisomorphic (as 3-ary trees) 3-ary trees with n vertices, and by convention set $t_0 = 1$.
   Show that $t_n = \sum_{i=0}^{n-1}(\sum_{j=0}^{n-1-i} t_i t_j t_{n-1-(i+j)})$.