Math 61-1 Final exam

[redacted]

TOTAL POINTS

70.5 / 90

QUESTION 1

Multiple choice 10 pts

1.1 **2 / 2**

✓ **- 0 pts** **Correct (c)**

- 2 pts Incorrect

1.2 **0 / 2**

- 0 pts Correct (a)

✓ **- 2 pts** **Incorrect**

1.3 **2 / 2**

✓ **- 0 pts** **Correct (c)**

- 2 pts Incorrect

1.4 **2 / 2**

✓ **- 0 pts** **Correct (b)**

- 2 pts incorrect

1.5 **2 / 2**

✓ **- 0 pts** **Correct (a)**

- 2 pts Incorrect

QUESTION 2

Short answer 10 pts

2.1 **1 / 2**

- 0 pts Correct ((-2)^100 + 3^100)

✓ **- 1 pts** **Almost correct (small arithmetic error in answer)**

- 2 pts Incorrect

2.2 **0 / 2**

- 0 pts Correct (C(7,4)6!4!)

- 1 pts Close

✓ **- 2 pts** **Incorrect**

2.3 **2 / 2**

✓ **- 0 pts** **Correct (24C4)**

- 1 pts Close

- 2 pts incorrect

2.4 **2 / 2**

✓ **- 0 pts** **Correct**

- 1 pts Close (Three of four)

- 2 pts Incorrect

2.5 **0 / 2**

- 0 pts Correct (2^(n^2 - n) + 2^(n^2 +n / 2) - 2^(n^2 - n / 2))

- 1 pts Close

✓ **- 2 pts** **Incorrect**

QUESTION 3

Equivalence relation 10 pts

3.1 it is an equivalence relation **4 / 4**

✓ **- 0 pts** **Correct**

- 1 pts issue in transitivity

- 3 pts misunderstanding of what relation is saying

- 4 pts blank

- 2 pts misunderstanding of symmetry

- 1 pts the decimal thing isn't exactly right, e.g. -.3 is related to .7

- 0 pts Click here to replace this description.

- 1 pts issue with symmetry

3.2 defining a function **2 / 4**

- 0 pts Correct

- 4 pts blank

- 2 pts need to prove uniqueness part of function

- 2 pts missing existence part of function

✓ **- 1 pts** **issue with uniqueness part of function**

✓ **- 1 pts need to consider different elements in the same equivalence class**

- 1 pts thing with decimals isn't quite right, for example -.3 and .7 are related

- 3 pts big misunderstanding of the equivalence relation or function

3.3 a function that doesn't descend **0 / 2**

- 0 pts Correct

✓ **- 2 pts your g is not a function**

- 1 pts issue with justification

- 1 pts your g does not work

- 2 pts blank

QUESTION 4

m-ary tree 10 pts

4.1 number of internal vertices **5 / 5**

✓ **- 0 pts Correct**

- 1 pts No/incorrect answer

- 4 pts No/incorrect justification

- 2 pts Didn't justify number of total vertices

- 3 pts "Proof by example"

- 2 pts Assumed every terminal vertex had the same height as the tree

- 5 pts Nothing

- 1 pts Forgot to account for root

- 2 pts Didn't subtract off internal vertices

4.2 height **4 / 5**

- 0 pts Correct

- 1 pts No base case

- 1 pts Didn't set up/invoke induction

✓ **- 1 pts Backwards inductive step (didn't show inductive construction is exhaustive)**

- 2 pts Compared to complete tree without showing this case is extremal

- 3 pts Assumed tree is complete / inductive construction forms complete trees from complete trees

- 1 pts Assumed all immediate subtrees have height h-1

- 4 pts "Proof by example"

- 5 pts Nothing shown / Incorrect reasoning

Define your variables!

QUESTION 5

spanning trees 10 pts

5.1 unique mst **3 / 6**

- 0 pts Correct

✓ **- 3 pts Appeal to Prim's or Kruskal's Algorithm (without proving it can generate any MST)**

- 6 pts No / Invalid reasoning

5.2 non unique spanning tree **4 / 4**

✓ **- 0 pts Correct**

- 4 pts Not an example

- 4 pts Claimed no such graph exists

- 4 pts Nothing

QUESTION 6

planar graphs 10 pts

6.1 2e > 3f **3 / 3**

✓ **+ 3 pts Correct**

+ 2 pts >= 3 edges for each face

+ 1 pts >= 3 edges for each face (w/ mistake)

+ 1 pts <=2 faces for each edge

+ 0 pts Incorrect

6.2 e<3v-6 **3 / 3**

✓ **+ 3 pts Correct**

+ 2 pts Euler's formula

+ 1 pts Correct application with (a)

+ 0 pts Incorrect

6.3 nonplanar graph **0 / 4**

+ 4 pts Correct

+ 3 pts Isomorphic to K_3,3

+ 2 pts Mistaken/missing ismorphism to K_3,3

+ 1 pts E <= 2v-4 or 2E >= 4F

+ 1 pts Other partial credit

✓ + **0 pts** **Incorrect**

QUESTION 7

10 pts

7.1 7^n-1 divisible by 6 **5 / 5**

✓ + **5 pts** **Correct**

+ **1 pts** Base case
+ **1 pts** Inductive hypothesis
+ **2 pts** factoring out a 7 in inductive step as (6+1) or adding/substracting 7
+ **1 pts** Conclusion
+ **0 pts** Incorrect

7.2 number with only 1s divisible by 7 **5 / 5**

✓ + **5 pts** **Correct**

+ **0 pts** Click here to replace this description.
+ **1 pts** Look at 8 consecutive terms
+ **1 pts** Pigeonhole remainder
+ **1 pts** 7 divides a number of the form 111..000...
+ **2 pts** This implies that 7 divides 10^k*11...
+ **1 pts** Unsuccessful attempt with substantial work

QUESTION 8

balanced binary trees 10 pts

8.1 **4 / 4**

✓ - **0 pts** **Correct**

- **2 pts** incomplete, need to describe how a height n minimal balanced binary tree is made out of ones of smaller height
- **3 pts** can't just do examples
- **4 pts** blank
- **1 pts** how are you adding in these trees/ vertices?
- **3 pts** can't do induction without using some properties of minimal balanced binary trees
- **4 pts** incorrect numbers/ equation

8.2 relationship to fibonacci numbers **3 / 3**

✓ - **0 pts** **Correct**

- **1.5 pts** that is not the recurrence/ equation for the fibonacci numbers/ minimal balanced binary trees
- **1 pts** you are assuming the desired conclusion
- **3 pts** blank
- **1.5 pts** need to use recurrence for fiboacci numbers
- **1.5 pts** missing inductive step
- **1 pts** the two recurrences aren't exactly the same, you need to account for this difference
- **0.5 pts** error in equations
- **1 pts** need to check initial conditions

8.3 Theta **2.5 / 3**

- **0 pts** Correct

✓ - **0.5 pts** **need to account for other term in equation for fibonacci numbers (sometimes it is contributing something positive, something something negative)**

- **2 pts** wrong formula for fibonacci numbers/ v_n
- **1 pts** issue with big O
- **1 pts** issue with omega
- **3 pts** blank/ no gradable work
- **1 pts** wrong equations/ issues with constants
- **2 pts** need to use equation for v_n/ Fibonacci numbers

QUESTION 9

binomial coefficients 10 pts

9.1 3^n **4 / 4**

✓ + **4 pts** **Correct**

+ **3 pts** Minor error
+ **2 pts** Binomial theorem
+ **1 pts** Attempted induction or counting argument
+ **0 pts** Incorret

9.2 vandermonde identity **6 / 6**

✓ + **6 pts** **Correct**

+ **5 pts** Minor errror
+ **3 pts** One part of counting argument or (x+y)^n+m
+ **1 pts** Attempted to use induction/binomial thrm/Pascal's identity
+ **0 pts** Incorrect

Final

Name: [redacted]

Student ID: [redacted]

Section:

	Tuesday:	Thursday:	
	1A	1B	TA: Albert Zheng
	1C	1D	TA: Benjamin Spitz
	1E	1F	TA: Eilon Reisin-Tzur

Instructions: Do not open this exam until instructed to do so. Please print your name and student ID number above, and circle the number of your discussion section. **You may not use calculators,** books, notes, or any other material to help you. Please make sure **your phone is silenced** and stowed where you cannot see it. Remember that you are bound by a conduct code.

Please get out your id and be ready to show it during the exam.

Please do not write below this line.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
Total:	90	

1. (10 points) Circle the correct answer (only one answer is correct for each question)

   1. $\frac{n!}{k!(n-k)!} + \frac{n!}{(k+1)!(n-k-1)!} =$

      (a) $\frac{(n+k)!}{k!n!}$

      (b) $\frac{(n+1)!}{k!(n+1-k)!}$

      (c) $\frac{(n+1)!}{(k+1)!(n-k)!}$

      (d) none of the above

   $\binom{n}{k} + \binom{n}{k+1}$

   $\frac{n!\,(k+1)}{(k+1)!\,(n-k)!} + \frac{(n-k)\,n!}{(k+1)!\,(n-k)!} =$

   $\frac{n!\,(n-k+k+1)}{(k+1)!\,(n-k)!}$

   $\frac{(n+1)!}{(k+1)!\,(n-k)}$

   2. The decision tree of a sorting algorithm for sorting n items (where at each step we can only decide whether or not one item is less than other) necessarily has:

      (a) a height of $\geq \lg(n!)$

      (b) a height of $\Omega \lg(n!)$ (but not necessarily a height of $\geq \lg(n!)$)

      (c) a height of $O(\lg(n!))$

      (d) a height of $O(n \lg n)$

   3. If G is a graph with n vertices and $n-2$ edges, then:

      (a) G is a tree

      (b) G is connected

      (c) G is disconnected

      (d) G is simple

Question 1 continues on the next page...

Question 1 continued...

4. Which of these graphs has an Euler cycle?
   (a) K_4
   (b) K_5
   (c) $K_{3,3}$
   (d) $K_{2,3}$

5. What is the *fewest* number of edges (i.e. in the best case) that could be examined by Dijkstra's algorithm on a graph with n vertices? (We examine edges in the part of the algorithm where we update labels.) You answer should be true for all n.
   (a) Less than or equal to n
   (b) More than n but less than or equal to $n^2/2$
   (c) More than $n^2/2$ but less than or equal to n^2
   (d) More than n^2

2. In this question write down your answer, no need for any justification. Leave your answers in a form involving factorials, $P(n,m)$, $\binom{n}{m}$, exponents, etc.

(a) (2 points) If $s_n = s_{n-1} + 6s_{n-2}$ and $s_0 = 2$, $s_1 = 1$; what is s_{100}?

$t^2 - t - 6 = 0$

$(t-3)(t+2)$

$S = b3^n + d(-2)^n$

$S_0 = 2 = b + d \quad b = 2 - d$

$1 = 3b - 2d$

$1 = 3(2-d) - 2d$

$1 = 6 - 3d - 2d$

$-5 = -5d$

$d = -1$

$b = 3$

$$S_{100} = 3(3)^{100} - 1(-2)^{100}$$

(b) (2 points) How many ways can 7 distinct math majors and 4 distinct CS majors sit in a circle, if the CS majors won't sit by each other and we say that two seatings are the same if they are related by a rotation?

$\frac{7!\,P(8,4)}{11}$

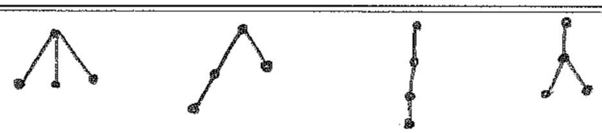
1-1-1-1-1-1-1-1

$$\frac{7!\,P(8,4)}{11}$$

(c) (2 points) A squirrel has 20 identical acorns that she is going to hide among 5 distinct holes. In how many ways can the squirrel hide the acorns?

$$\binom{24}{4}$$

(d) (2 points) Draw all the distinct (up to isomorphism) rooted trees with 4 vertices. Please put the root at the top.



(e) (2 points) What is the number of relations that are symmetric or reflexive on a set with n-elements?

$$2^{\frac{n^2-n}{2}}$$

Set w/ n elements

n^2 possible connections

2^{n^2-n} reflexive symmetric means every one has to have other

n^2-n for reflexive

$2^{\frac{n^2-n}{2}}$

3. Consider the relation on the real numbers defined by $C = \{(x, y) \in \mathbb{R}\times\mathbb{R} : x - y \in \mathbb{Z}\}$.

(a) (4 points) Show that C is an equivalence relation.

equivalence relation $\Rightarrow$ reflexive, symmetric, transitive

reflexive: If we have $x\,C\,y$ s.t. $x = y$ this means $x - y = 0$ which is contained in $\mathbb{Z}$. Thus, for every $(x,y) \in \mathbb{R}\times\mathbb{R}$ if $x = y$, then $x\,C\,y$

Symmetric: Symmetry means for $x\,C\,y$, then $y\,C\,x$

Suppose we know for arbitrary numbers x, y that $x\,C\,y$ this means that $x - y$ is an integer. The negative of an integer is still an integer, thus $-(x-y) \in \mathbb{Z}$
$\Rightarrow y - x \in \mathbb{Z}$ and $y\,C\,x$

transitive: Suppose we have $x\,C\,y$ and $y\,C\,z$ for $(x,y) \in \mathbb{R}\times\mathbb{R}$ and $(y,z) \in \mathbb{R}\times\mathbb{R}$. this means $x - y \in \mathbb{Z}$ and $y - z \in \mathbb{Z}$ and we want to show $x - z \in \mathbb{Z}$. We see that $x - z$ is simply $(x-y)+(y-z)$ and we know the sum of two integers is an integer. Thus, $x - z$ is an integer, $x - z \in \mathbb{Z}$ and $x\,C\,z$.

Question 3 continues on the next page. . .

(b) (4 points) Let $\tilde{\mathbb{R}}$ denote the set of equivalence classes of C, i.e. $\tilde{\mathbb{R}} = \{[x] : x \in \mathbb{R}\}$. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x + 1/2$.

Show that the relatation $\tilde{f}$ from $\tilde{\mathbb{R}}$ to $\tilde{\mathbb{R}}$ defined by $\tilde{f} = \{([a],[b]) \in \tilde{\mathbb{R}} \times \tilde{\mathbb{R}} : f(a) = b\}$ is a function.

To prove that $\tilde{f}$ is a function, we must show that every input has a single output. In the case of $\tilde{f}$ the inputs and outputs are the equivalence classes of C. For a number a, if we take $a + \frac{1}{2} = b$, we need to show that b only belongs to one equivalence class of C. By definition of equivalence classes, every element $x \in \mathbb{R}$ belongs to one equivalence class defined by C. In addition, we need to show that for every input a into f(x), there is only one output b. Since f(x) simply takes the number and adds a half to it, we know that every input will produce only one output since there is no way to generate multiple unique numbers from a single input from adding a constant to that input. Thus since every input $a \in \mathbb{R}$ into f(x) produces a unique output b and every output $b \in \mathbb{R}$ is put into a single equivalence class defined by the relation C, every input [a] will produces a single output [b].

(c) (2 points) Give an example of a function $g : \mathbb{R} \to \mathbb{R}$ so that the relation $\tilde{g}$ from $\tilde{\mathbb{R}}$ to $\tilde{\mathbb{R}}$ defined by $\tilde{g} = \{([a],[b]) \in \tilde{\mathbb{R}} \times \tilde{\mathbb{R}} : g(a) = b\}$ is **not** a function. (Be sure to justify your answer.)

$g(x) = \frac{1}{x}$

for x=0 produces $g(0) = \infty \Rightarrow$ no equivalence class

4. For m a positive integer, a *full m-ary tree* is a rooted tree where every parent has exactly m children.

(a) (5 points) If T is a full m-ary tree with i internal vertices, how many terminal vertices does T have?

$mi+1-i$

Every internal vertex has m children, thus there are mi vertices not including the root. There are $mi+1$ total vertices and i internal vertices. Thus there are $mi+1-i$ terminal vertices

(b) (5 points) Show that if T is a full m-ary tree of height h with t terminal vertices, then $t \leq m^h$.

induction

base case $h=0$ at zero height there is simply the root node

so $t=1$ $\quad m^0 = 1 \geq t$ ✓

assume

$t_h \leq m^h$

we want to show

$t_{h+1} \leq m^{h+1}$

suppose we have a m-ary tree of height h and want to increase the height of the tree by 1. To do this, we add a root node to the top. The maximum number of terminal vertices we can add is $m t_h$ because we are adding a subtree of height h to all m nodes at height level 1. since $t_h \leq m^h$ we can rewrite

$t_{h+1}max = m t_h \leq m^{h+1}$ $\quad t_{h+1} \leq m(m^h) \leq m^{h+1}$

$t_{h+1} \leq m^{h+1}$

Thus, by induction, $t \leq m^h$. Note that the inequality comes from the fact that the maximum terminal vertices for t_{h+1} is m^{h+1} but that there can be less than that number of terminal vertices

5. (a) (6 points) Show that if G is a connected weighted graph where all the edges of G have distinct weights then G has a unique minimal spanning tree.

First, we know that G contains a spanning tree within it because G is connected. We now attempt to show that if G has unique weights for each edge that G has a unique minimal spanning tree. Let us construct the spanning tree using Prim's algorithm which produces a minimal spanning tree of a graph (which we proved in class) and attempt to argue that the constructed spanning tree is unique to the graph. Picking an arbitrary vertex in G, we examine all edges adjacent to it. Since all the edges of G have distinct weights, we know that there will be one edge that is the minimum of all the edges adjacent to it. We pick this edge and add it to the tree and the vertex it is connected to to the tree. We continue this way for the rest of the graph and see a resulting minimal spanning tree. We now show that this spanning tree is unique to G. Since the algorithm continuously searches for the minimum edge adjacent to obtained vertices and G has distinct weights, the edges chosen by the algorithm will always be the minimal edge of the graph and thus there is a unique minimal spanning tree for each G

case when vertex not yet reached and:

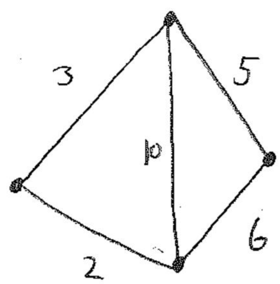
degree 1 — must chose that path

degree 2 — algorithm will chose smaller path

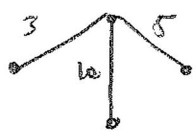
degree 3 — algorithm will chose smaller path

Question 5 continues on the next page…

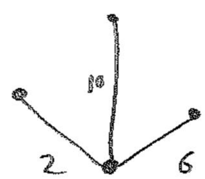
(b) (4 points) Give an example of a connected weighted graph G so that all the edges of G have distinct weights and G has at least two distinct spanning trees that have the same total weight, i.e. the sums of the weights of the edges in these two distinct trees agree, *or* prove that no such weighted graph exists.



distinct weights



$3+5+10 = 18$



$2+6+10 = 18$

6. (a) (3 points) Show that for G a connected simple planar graph containing a cycle if G has E edges and F faces, then $2E \geq 3F$.

Since G is a connected simple planar graph, the minimum number of edges needed to make a face is 3 because there are no parallel edges (which would make a face with 2) or loops (which would make a face with one). We know that each edge can be used on both sides to contribute to making a face. We also know that each face has to be comprised of at least 3 edges. Thus $2E$ (the total number of components that can be used to make a face) $\geq 3F$ (the number of faces times the minimum number of components used to make a face. The inequality is present because some faces may take more than 3 "components" to be made.

(b) (3 points) Show that for G a connected simple planar graph containing a cycle if G has E edges and V vertices, then $E \leq 3V - 6$.

We established in the previous part that for a connected simple planar graph, $2E \geq 3F$. Since G is a planar graph, then $F = E - V + 2$

plugging that into the above inequality

$$2E \geq 3(E - V + 2)$$

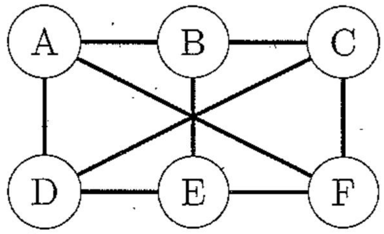
$$2E \geq 3E - 3V + 6$$

$$3V - 6 \geq E$$

$$E \leq 3V - 6 \checkmark$$

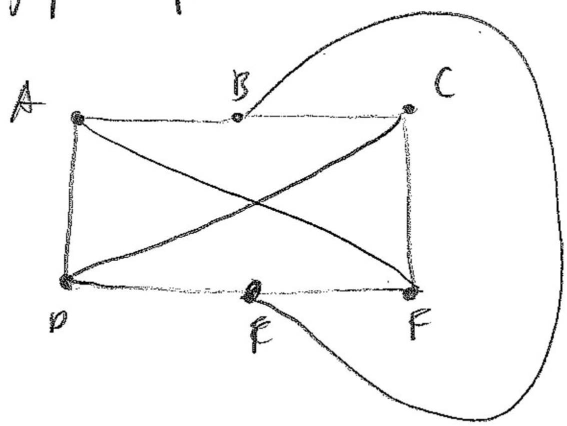
Question 6 continues on the next page...

(c) (4 points) Is the following graph planar? If it is give a planar drawing of it. If not, prove that it is not planar.



$E=9$
$V=6$

This graph is planar



7. (a) (5 points) Show that for all $n \geq 1$, $7^n - 1$ is divisible by 6.

Induction

base case $n=1$ $\quad 7^1 - 1 = 6 \quad$ 6 is divisible by 6 ✓

Induction assume $7^n - 1 \bmod 6 = 0$

show $7^{n+1} - 1 \bmod 6 = 0$

$$7^{n+1} - 1 = 7 \cdot 7^n - 1 = 6 \cdot 7^n + 7^n - 1$$

Since $7^n - 1$ mod 6 is 0, we only need to consider the term $6 \cdot 7^n$. Since $6 \cdot 7^n$ is obviously divisible by 6, we can conclude that the whole term $6 \cdot 7^n + 7^n - 1$ mod 6 is zero and thus $7^{n+1} - 1$ mod $6 = 0$. Thus, we are done by induction

(b) (5 points) Show that there is a number of the form $\sum_{i=0}^{n} 10^i$ (i.e. a number consisting only of 1s) that is divisible by 7.

The question above produces numbers of the form

1, 11, 111, 1111, 11111, 1111111, 11111111, etc.

For all these numbers, if we divide each by 7, there will be a remainder ranging from 0-6 => 7 possible remainders.

By pigeonhole principle, there are 7 possible remainders and $n > 7$ numbers that we are considering so there will thus be 2 numbers with the same remainder. Subtracting these numbers with each other (bigger number minus smaller number) will produce

$$\underbrace{11111\ldots}_{m\ 1s} - \underbrace{11\ldots}_{n\ 1s} = \underbrace{11111\ldots}_{m-n\ 1s} \cdot 10^x$$

Since these two numbers subtracted by 7 have the same remainder, the difference of these numbers is divisible by 7 ($x \bmod 7 - x \bmod 7 = 0 \bmod 7$). Furthermore, since any power of 10 is not divisible by 7, the $\underbrace{111\ldots}_{m-n\ 1s}$ term must be divisible by 7. Thus, there is a number of form 1111... that is divisible by 7.

8. A *balanced binary tree* is a binary tree where for each vertex the heights of the left and right subtrees of that vertex differ by at most one. Let v_n denote the minimum number of vertices in a balanced binary tree of height n.

(a) (4 points) Show that v_n satisfies for $n \geq 2$ the recurrence $v_n = v_{n-1} + v_{n-2} + 1$

Induction

base case: $V_2 = 4$, $V_3 = 7$, $V_4 = 12$ ✓

WTS $V_{n+1} = V_n + V_{n-1} + 1$

assume $V_n = V_{n-1} + V_{n-2} + 1$. Assume we have a balanced minimal binary tree of height n and want to add another level. To do this we add a root node at the top of V_n. However, we must have the left and right subtrees of the new root node differ by at most one. Thus, we must add another subtree of the new root of height $n-1$. Now we have the root node plus the subtree of height n with V_n vertices and another subtree of height $n-1$ with V_{n-1} vertices on the other child. Thus $V_{n+1} = V_n + V_{n-1} + 1$. Since we know V_n and V_{n-1} are the minimal balanced binary trees of height n and $n-1$, respectively we know V_{n+1} is the minimal and we are done.

(b) (3 points) Show that for $n \geq 0$, $v_n = \cancel{F_{n+2}}$ $F_{n+3} - 1$, where F_k is the k^{th} Fibonacci number.

Induction $F_0 = 0$ $F_1 = 1$ $F_2 = 1$ $F_3 = 2$

base case $n = 0$ $V_0 = F_3 = 1$ ✓

assume $V_n = F_{n+3} - 1$ WTS $V_{n+1} = F_{n+4} - 1$

$V_{n+1} = F_{n+4} - 1$ from part a) we know $V_{n+1} = V_n + V_{n-1} + 1$

$V_n + V_{n-1} + 1 = F_{n+4} - 1$ and we know $V_n = F_{n+3} - 1$ and $V_{n-1} = F_{n+2} - 1$

$F_{n+3} - 1 + F_{n+2} - 1 + 1 = F_{n+4} - 1$ $\Rightarrow$ $F_{n+4} = F_{n+2} + F_{n+3}$

Since this is the formula for Fibonacci numbers we know it is true and thus $V_{n+1} = F_{n+4} - 1$. By induction,

$V_n = F_{n+3} - 1$

Question 8 continues on the next page...

$\frac{\sqrt{5}}{\sqrt{5}} \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)$ $\frac{\sqrt{5}+5}{10}$

$\frac{\sqrt{5}}{5}\left(\frac{1+\sqrt{5}}{2}\right)$

(c) (3 points) Show that $v_n = \Theta(\phi^{n+2})$, where $\phi = \frac{1+\sqrt{5}}{2}$.

$V_n = \Theta\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+2}\right)$

we showed earlier that $V_n = F_{n+3} - 1$

Let us solve the recurrence relation

$F_{n+3} = F_{n+2} + F_{n+1}$ $\qquad \frac{1 \pm \sqrt{1-4(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$

$F_{n+3} - F_{n+2} - F_{n+1} = 0$

$t^2 - t - 1 = 0$ $\qquad S = b\left(\frac{1+\sqrt{5}}{2}\right)^n + d\left(\frac{1-\sqrt{5}}{2}\right)^n$

$F_0 = 0 \quad F_1 = 1$

$b = -d$

$0 = b + d$ $\qquad 1 = \frac{b + b\sqrt{5}}{2} - \frac{b + \sqrt{5}}{2}$

$1 = b\left(\frac{1+\sqrt{5}}{2}\right) + d\left(\frac{1-\sqrt{5}}{2}\right)$ $\qquad 1 = b\sqrt{5} \quad b = \frac{1}{\sqrt{5}} \quad d = -\frac{1}{\sqrt{5}}$

$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$

$\frac{1-\sqrt{5}}{2} \leq -1$

thus $V_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+3} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+3} - 1$ $\qquad$ we know that $F_{n+1} \geq 0$

$V_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+2} + \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1} - 1$

?

$\leq \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - \frac{1}{\sqrt{5}}(-1)\left(\frac{1-\sqrt{5}}{2}\right)^{n+1} + \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$

$\leq \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} + \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}$

$\leq \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} + \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} \leq \frac{2}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} \leq 2\left(\frac{1+\sqrt{5}}{2}\right)^{n+2}$

$\hookrightarrow V_n = O(\phi^{n+2})$

$V_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+3} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+3} - 1$

$(F_{n+3} - 1 \geq F_{n+2}) \rightarrow \geq \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+2}$ for n odd

$\geq \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+2}$

for n even

$\geq \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+3} \geq \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} \longrightarrow V_n = \Omega(\phi^{n+2})$

$\hookrightarrow \underline{V_n = \Theta(\phi^{n+2})}$

9. (a) (4 points) Show that $\sum_{i=0}^{n} 2^i \binom{n}{i} = 3^n$.

$$\sum_{i=0}^{n} 2^i \binom{n}{i} = 3^n$$

From the binomial theorem

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$$

Substitute $a = 2$, $b = 1$

$$(2+1)^n = \sum_{i=0}^{n} \binom{n}{i} 2^i 1^{n-i}$$

$$3^n = \sum_{i=0}^{n} \binom{n}{i} 2^i \quad \square$$

(b) (6 points) Show that $\binom{n+m}{r} = \sum_{i=0}^{r} \binom{n}{i}\binom{m}{r-i}$.

I have n+m elements and I want to choose r unique elements from them. Consider the case that I want to choose 0 elements from n and all r elements from m. There are $\binom{n}{0}\binom{m}{r}$ ways to do so. Consider the case that I want to choose 1 element from n and r-1 elements from m. There are $\binom{n}{1}\binom{m}{r-1}$ ways to do so. Consider the case that I want to choose 2 elements from n and r-2 elements from m. There are $\binom{n}{2}\binom{m}{r-2}$ ways to do so. I continue to increment the number I choose from n and decrement the number I choose from m by one until I have all possible combination of elements I chose from n and numbers I chose from m. I then add all these possibilities up and get the number of ways to choose r elements from m+n elements and get the above formula.

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