21W-MATH61-1 final exam

ERIC YANG

TOTAL POINTS

85 / 90

QUESTION 1

10 pts

1.1 2 / 2

✓ - **0 pts** Correct

1.2 2 / 2

✓ - **0 pts** Correct

1.3 2 / 2

✓ - **0 pts** Correct

1.4 2 / 2

✓ - **0 pts** Correct

1.5 2 / 2

✓ - **0 pts** Correct 23 or 24 (if it is required that the maximum be unique)

QUESTION 2

10 pts

2.1 3 / 3

✓ - **0 pts** Correct: $$2^{n^2 - n}$$

2.2 4 / 4

✓ - **0 pts** Correct: $$2^n 3^{\binom{n}{2}}$$

2.3 3 / 3

✓ - **0 pts** Correct: $$2^{n^2 - n} + 2^n 3^{\binom{n}{2}} - 3^{\binom{n}{2}}$$

QUESTION 3

10 pts

3.1 4 / 4

✓ - **0 pts** Correct

3.2 3 / 3

✓ - **0 pts** Correct

1 Partitions aren't ordered, so be careful talking about them like this as it can lead to mistakes.

3.3 3 / 3

✓ - **0 pts** Correct

QUESTION 4

10 pts

4.1 5 / 5

✓ - **0 pts** Correct

4.2 5 / 5

✓ - **0 pts** Correct

QUESTION 5

10 pts

5.1 2 / 2

✓ - **0 pts** Correct

5.2 2 / 2

✓ - **0 pts** Correct: $$n2^{n-1}$$

5.3 2 / 2

✓ - **0 pts** Correct: $$n2^{n-1}-2^n+2$$

5.4 4 / 4

✓ - **0 pts** Correct

QUESTION 6

10 pts

6.1 5 / 5

✓ **- 0 pts** Correct

6.2 5 / 5

✓ **- 0 pts** Correct

QUESTION 7

10 pts

7.1 5 / 5

✓ **- 0 pts** Correct

7.2 5 / 5

✓ **- 0 pts** Correct

QUESTION 8

10 pts

8.1 3 / 5

✓ **- 2 pts doesn't justify/ mistake in justifying why left and right subtrees each have 2^{h-1} terminal vertices / height $h-1$**

8.2 3 / 5

✓ **- 2 pts gives an isomorphism or good ideas for an isomorphism, but doesn't prove that it is an isomorphism.**

QUESTION 9

10 pts

9.1 4 / 5

✓ **- 1 pts only does one direction of the bijection/ doesn't argue why it is 1-1 and onto**

9.2 5 / 5

✓ **- 0 pts** Correct

Eric Yang

Math 61 Final

1a) 2

1b) 286

1c) $2^{20} = 1048576$

1d) $\{(a,a), (b,b), (a,b), (b,a), (c,c), (d,d), (c,d), (d,c), (e,e)\}$

1e) 23

1.1 2 / 2

✓ **- 0 pts** Correct

Eric Yang

Math 61 Final

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1b) 286

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1d) $\{(a,a), (b,b), (a,b), (b,a), (c,c), (d,d), (c,d), (d,c), (e,e)\}$

1c) 23

1.2 2 / 2

✓ **- 0 pts** Correct

Eric Yang

Math 61 Final

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1c) 23

1.3 **2 / 2**

✓ **- 0 pts** Correct

Eric Yang

Math 61 Final

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Math 61 Final

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1e) 23

1.5 2 / 2

✓ **- 0 pts** **Correct 23 or 24 (if it is required that the maximum be unique)**

2a) We know for a reflexive relation every element must be related to itself, making n pairs that have to be included. If there are n^2 pairs total that means there are n^2-n pairs left that we can include or not. Using $\sum_{i=0}^{n^2-n} C\binom{n^2-n}{i}$ we (binomial theorem) find there are 2^{n^2-n} reflexive relations on X with n elements.

2b) An atisymmetric relation R implies if aRb and bRa then $a=b$. There are n pairs that are in the form of (a,a) where $a \in X$. These pairs can either be included or not, giving 2^n possibilities. For n^2-n pairs in the form of (a,b) where $b \neq a$, either (a,b), (b,a), or none can be included. However, the n^2-n pairs include both (a,b) and (b,a), meaing there are only $\frac{n^2-n}{2}$ pairs that have the above 3 possibilities. This gives $3^{\frac{n^2-n}{2}}$ possible outcomes. Using multiplication principle we get $2^n 3^{\frac{n^2-n}{2}}$ total possibilities.

2c) Using inclusion-exclusion principle, we know that the total number of antisymmetric or reflexive relations = number of reflexive relations + number of antisymmetric relations − number of antisymmetric and reflexive relations. We can find the number of antisymmetric and reflexive pairs by taking our answer from part b and dividing by 2^n since the n pairs in the form (a,a) where $a \in X$ (or otherwise the diagonal of the matricies of relations) must be included now where previously they could either be included or not. This yields us with $3^{\frac{n^2-n}{2}}$ reflexive and antisymmetric relations, in turn giving us $2^{n^2-n} + 2^n 3^{\left(\frac{n^2-n}{2}\right)} - 3^{\left(\frac{n^2-n}{2}\right)}$ reflexive or antisymmetric relations

$= 2^{n^2-n} + (2^n-1)\, 3^{\left(\frac{n^2-n}{2}\right)}$

2.1 3 / 3

✓ **- 0 pts** **Correct: $$2^{n^2 - n}$$**

2a) We know for a reflexive relation every element must be related to itself, making n pairs that have to be included. If there are n^2 pairs total that means there are n^2-n pairs left that we can include or not. Using $\sum_{i=0}^{n^2-n} C\binom{n^2-n}{i}$ we (binomial theorem) find there are 2^{n^2-n} reflexive relations on X with n elements.

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$= 2^{n^2-n} + (2^n-1)3^{\left(\frac{n^2-n}{2}\right)}$

2.2 4 / 4

✓ **- 0 pts** **Correct: $$2^n 3^{\binom{n}{2}}$$**

2a) We know for a reflexive relation every element must be related to itself, making n pairs that have to be included. If there are n^2 pairs total that means there are $n^2 - n$ pairs left that we can include or not. Using $\sum_{i=0}^{n^2-n} C\binom{n^2-n}{i}$ we (binomial theorem) find there are 2^{n^2-n} reflexive relations on X with n elements.

2b) An atisymmetric relation R implies if aRb and bRa then $a=b$. There are n pairs that are in the form of (a,a) where $a \in X$. These pairs can either be included or not, giving 2^n possibilities. For n^2-n pairs in the form of (a,b) where $b \neq a$, either (a,b), (b,a), or none can be included. However, the n^2-n pairs include both (a,b) and (b,a), meaing there are only $\frac{n^2-n}{2}$ pairs that have the above 3 possibilities. This gives $3^{\frac{n^2-n}{2}}$ possible outcomes. Using multiplication principle we get $2^n 3^{\frac{n^2-n}{2}}$ total possibilities.

2c) Using inclusion - exclusion principle, we know that the total number of antisymmetric or reflexive relations = number of reflexive relations + number of antisymmetric relations - number of antisymmetric and reflexive relations. We can find the number of antisymmetric and reflexive pairs by taking our answer from part b and dividing by 2^n since the n pairs in the form (a,a) where $a \in X$ (or otherwise the diagonal of the matricles of relations) must be included now where previously they could either be included or not. This yields us with $3^{\frac{n^2-n}{2}}$ reflexive and antisymmetric relations, in turn giving us
$2^{n^2-n} + 2^n 3^{\left(\frac{n^2-n}{2}\right)} - 3^{\left(\frac{n^2-n}{2}\right)}$ reflexive or antisymmetric relations
$= 2^{n^2-n} + (2^n - 1)\, 3^{\left(\frac{n^2-n}{2}\right)}$

2.3 3 / 3

✓ **- 0 pts** **Correct: $$2^{n^2 - n} + 2^n 3^{\binom{n}{2}} - 3^{\binom{n}{2}}$$**

Proof by induction:

3a) Base case: $n = 0 \Rightarrow$ We know that $B_0 = 1$ since the only way to partition 0 elements is just an empty set. (also given)

Inductive step: Assume we know that $B_0, B_1, \dots B_{n-1}$ are the number of partitions of $0, 1, \dots n-1$ elements respectively. Going from $n-1$ elements to n elements, we add a new element. Let's call it X. X can either be in a group by itself or with up to $n-1$ elements (the other elements in the set). Thus, there are $\sum_{k=0}^{n-1} \binom{n-1}{k}$ ways to choose up to $n-1$ elements to group with X where k represents the number of elements grouped with X. Removing all the elements grouped with X, we are left with $n-1-k$ elements. Since we know the number of ways to petition $0, 1 \dots n-1$ elements as $B_0, B_1, \dots B_{n-1}$ respectively, the total number of ways to partition all n elements $= \sum_{k=0}^{n-1} \binom{n-1}{k} B_{n-1-k}$. (we multiply $\sum_{k=0}^{n-1} \binom{n-1}{k}$ with B_{n-1-k} because of multiplication principle.) Since we have proven $B_n = \sum_{k=0}^{n-1} \binom{n-1}{k} B_{n-1-k}$ the inductive step is done. Using the base case and inductive step, proof by induction tells us $B_n = \sum_{k=0}^{n-1} \binom{n-1}{k} B_{n-1-k}$ for all $n > 0$.

3b) For $n = 1$, $2^{1-1} = 1$ and $B_n = 1$, thus $2^{n-1} \leq B_n$ is true. As n increases, 2^{n-1} doubles for each iteration of n. For B_n, if we were to add an additional element, there are at least 2 possible cases. First, if we were to just add an additional group to each of the B_{n-1} partitions that just included the new element, this would yield us with B_{n-1} partitions of n elements. Additionally, another possibility is that we can add the new element to the first group of each of the B_{n-1} partitions (we know each partition has at least 1 group). This would give us another B_{n-1} partitions of n elements. We could continue this for all partitions of $n-1$ elements where there are more than 1 group.

3.1 4 / 4

✓ **- 0 pts** Correct

Proof by induction:

3a) Base case: $n=0 \Rightarrow$ We know that $B_0 = 1$ since the only way to partition 0 elements is just an empty set. (also given)

Inductive step: Assume we know that $B_0, B_1, \ldots B_{n-1}$ are the number of partitions of $0, 1, \ldots n-1$ elements respectively. Going from $n-1$ elements to n elements, we add a new element. Let's call it X. X can either be in a group by itself or with up to $n-1$ elements (the other elements in the set). Thus, there are $\sum_{k=0}^{n-1} \binom{n-1}{k}$ ways to choose up to $n-1$ elements to group with X where k represents the number of elements grouped with X. Removing all the elements grouped with X, we are left with $n-1-k$ elements. Since we know the number of ways to petition $0, 1 \ldots n-1$ elements as $B_0, B_1, \ldots B_{n-1}$ respectively, the total number of ways to partition all n elements $= \sum_{k=0}^{n-1} \binom{n-1}{k} B_{n-1-k}$. (we multiply $\sum_{k=0}^{n-1} \binom{n-1}{k}$ with B_{n-1-k} because of multiplication principle.) Since we have proven $B_n = \sum_{k=0}^{n-1} \binom{n-1}{k} B_{n-1-k}$ the inductive step is done. Using the base case and inductive step, proof by induction tells us $B_n = \sum_{k=0}^{n-1} \binom{n-1}{k} B_{n-1-k}$ for all $n>0$.

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implying that $B_n \geq 2B_{n-1}$, or that B_n at least doubles every iteration of n. This tells us that B_n at least grows as fast as 2^n, making $2^{n-1} \leq B_n$ for all $n \geq 1$.

3c) 2^{n^2} is equal to the total number of relations on a set of n elements since each pair can either be included or not. We also know that the number of partitions of n elements is equal to the number of equivalence relations on the set since there exists a bijection between partitions on a set and equivalence relations on the same set. The number of equivalence relations is at most the total number of relations on the set (2^{n^2}) for all $n \geq 1$. Therefore, the total number of partitions of the set (which is equal to the total number of equivalence relations) must also not exceed the total number of relations, giving us $B_n \leq 2^{n^2}$ for all $n \geq 1$.

3.2 3 / 3

✓ **- 0 pts** Correct

1 Partitions aren't ordered, so be careful talking about them like this as it can lead to mistakes.

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3.3 3 / 3

✓ **- 0 pts** Correct

4a) We know that there are $\binom{n}{2}$ edges in K_n since each vertex is connected to every other vertex. A subgraph of K_n that contains all n verticles still means that only edges can be removed. For each edge, it can either be in the subgraph or not, giving us $2^{\binom{n}{2}}$ possible subgraphs. (multiplication principle)

4b) We know that if we are to select m verticles from K_n and the edges between those vertices, we have K_m, since there are m vertices and each vertex is connected to every other vertex. We also know that the total number of subgraphs for n vertices where all vertices of K_n must be contained is $2^{\binom{n}{2}}$ (from part a). Thus, we know that for K_m, the total number of subgraphs that still contain all m vertices is $2^{\binom{m}{2}}$. We can find the total number of subgraphs of K_n by summing all subgraphs of K_m where $0 \leq m \leq n$. However, there are also $\binom{n}{m}$ ways to select m vertices from K_n. Thus, the total number of subgraphs of $K_n = \sum_{m=0}^{n} \binom{n}{m} 2^{\binom{m}{2}}$.

4.1 5 / 5

✓ **- 0 pts** Correct

4a) We know that there are $\binom{n}{2}$ edges in K_n since each vertex is connected to every other vertex. A subgraph of K_n that contains all n verticles still means that only edges can be removed. For each edge, it can either be in the subgraph or not, giving us $2^{\binom{n}{2}}$ possible subgraphs. (multiplication principle)

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4.2 5 / 5

✓ **- 0 pts** Correct

5a) Since there is no edge that connects from a vertex to itself in an n-cube, the shortest way to get back to a vertex is to change a bit and then change it back. However, this is not counted as a cycle since the same edge cannot be repeated. Therefore, the shortest path must involve 2 bits. In order to return to a vertex, each bit has to be inversed at least twice or an even number of times to reach the original state of the bit. Thus, the shortest cycle that changes 2 bit consists of 4 inversions (which each are represented by an edge), making the length of every cycle at least 4 (the length of the shortest cycle).

5b) An n-cube contains 2^n vertices, since each vertex can be represented by a bit string of length n and each bit has 2 possibilities. Each vertex is connected to n edges since each bit can be inverted. However, each edge connects 2 vertices. Using multiplication principle we find that there are $(2^n \cdot n)/2$ or $2^{n-1} \cdot n$ edges in an n-cube.

e = # of edges
v = # of vertices
f = # of faces

5c) Euler's formula tells us that $f = e - v + 2$. If Plugging in $2^{n-1}n$ for the number of edges and 2^n for the number of vertices (both derived above) give us that there are $2^{n-1}n - 2^n + 2$ faces in an n-cube.

5d) We know each face is bounded by at least 4 edges and each edge is part of 2 faces. Thus $2e \geq 4f$ for a planar graph. Plugging in our derived numbers from above, we get $2(2^{n-1}n) = 2^n n$ for left side and $4(2^{n-1}n - 2^n + 2)$ for the right side. This simplifies to $2^n(4-n) \geq 8$. For $n = 4$, $2^{(4)}(4-4) = 0 \geq 8$ is false, meaning 4-cube is not planar. As n increase the left will become negative, making all n-cubes where $n \geq 4$ not planar.

5.1 2 / 2

✓ **- 0 pts** Correct

5a) Since there is no edge that connects from a vertex to itself in an n-cube, the shortest way to get back to a vertex is to change a bit and then change it back. However, this is not counted as a cycle since the same edge cannot be repeated. Therefore, the shortest path must involve 2 bits. In order to return to a vertex, each bit has to be inversed at least twice or an even number of times to reach the original state of the bit. Thus, the shortest cycle that changes 2 bit consists of 4 inversions (which each are represented by an edge), making the length of every cycle at least 4 (the length of the shortest cycle).

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5.2 2 / 2

✓ **- 0 pts** **Correct: $$n2^{n-1}$$**

5a) Since there is no edge that connects from a vertex to itself in an n-cube, the shortest way to get back to a vertex is to change a bit and then change it back. However, this is not counted as a cycle since the same edge cannot be repeated. Therefore, the shortest path must involve 2 bits. In order to return to a vertex, each bit has to be inversed at least twice or an even number of times to reach the original state of the bit. Thus, the shortest cycle that changes 2 bit consists of 4 inversions (which each are represented by an edge), making the length of every cycle at least 4 (the length of the shortest cycle).

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5.3 2 / 2

✓ **- 0 pts Correct: $$n2^{n-1}-2^n+2$$**

5a) Since there is no edge that connects from a vertex to itself in an n-cube, the shortest way to get back to a vertex is to change a bit and then change it back. However, this is not counted as a cycle since the same edge cannot be repeated. Therefore, the shortest path must involve 2 bits. In order to return to a vertex, each bit has to be inversed at least twice or an even number of times to reach the original state of the bit. Thus, the shortest cycle that changes 2 bit consists of 4 inversions (which each are represented by an edge), making the length of every cycle at least 4 (the length of the shortest cycle).

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5c) Euler's formula tells us that $f = e - v + 2$. Plugging in $2^{n-1}n$ for the number of edges and 2^n for the number of vertices (both derived above) give us that there are $2^{n-1}n - 2^n + 2$ faces in an n-cube.

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5.4 4 / 4

✓ **- 0 pts** Correct

Proof by induction:

6a) Base case: $n=1$ $\quad \sum_{i=1}^{1} i^3 = 1 = \left(\frac{1(1+1)}{2}\right)^2$

Inductive step: Assume $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2 = \frac{n^2(n+1)^2}{4}$

We know that $\sum_{i=1}^{n+1} i^3 = (n+1)^3 + \sum_{i=1}^{n} i^3$

$$
\begin{aligned}
&= (n+1)^3 + \frac{n^2(n+1)^2}{4} \\
&= \frac{4n^3+12n^2+12n+4}{4} + \frac{n^4+2n^3+n^2}{4} \\
&= \frac{n^4+6n^3+13n^2+12n+4}{4} \\
&= \frac{(n+1)^2(n+2)^2}{4} \\
&= \frac{(n+1)^2((n+1)+1)^2}{4} \\
&= \left(\frac{(n+1)((n+1)+1)}{2}\right)^2
\end{aligned}
$$

$\Rightarrow$ the inductive step is complete and we have proven that $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$ by induction.

6b) Knowing that f is onto, this means there exists at least 1 $x \in X$ such that $f(x) = y$ for all $y \in Y$. Thus, we know that for $f \circ g = id_Y$, all we need is for g to map all $y \in Y$ to an $x \in X$ such that $f(x) = y$. We proved this is possible since there is at least 1 value of $x \in X$ that maps to every value of $y \in Y$. If there are multiple $x \in X$ such that $f(x) = y$, g only has to map y to one of those x values arbitrarily since g still has to be a function.

6.1 5 / 5

✓ **- 0 pts** Correct

Proof by induction:

6a) Base case: $n=1$ $\sum_{i=1}^{1} i^3 = 1 = \left(\frac{1(1+1)}{2}\right)^2$

Inductive step: Assume $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2 = \frac{n^2(n+1)^2}{4}$

We know that $\sum_{i=1}^{n+1} i^3 = (n+1)^3 + \sum_{i=1}^{n} i^3$

$$= (n+1)^3 + \frac{n^2(n+1)^2}{4}$$

$$= \frac{4n^3+12n^2+12n+4}{4} + \frac{n^4+2n^3+n^2}{4}$$

$$= \frac{n^4+6n^3+13n^2+12n+4}{4}$$

$$= \frac{(n+1)^2(n+2)^2}{4}$$

$$= \frac{(n+1)^2((n+1)+1)^2}{4}$$

$$= \left(\frac{(n+1)((n+1)+1)}{2}\right)^2$$

$\Rightarrow$ the inductive step is complete and we have proven that $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$ by induction.

6b) Knowing that f is onto, this means there exists at least 1 $x \in X$ such that $f(x) = y$ for all $y \in Y$. Thus, we know that for $f \circ g = id_Y$, all we need is for g to map all $y \in Y$ to an $x \in X$ such that $f(x) = y$. We proved this is possible since there is at least 1 value of $x \in X$ that maps to every value of $y \in Y$. If there are multiple $x \in X$ such that $f(x) = y$, g only has to map y to one of those x values arbitrarily since g still has to be a function.

6.2 5 / 5

✓ **- 0 pts** Correct

Proof by contradiction

7a) Assume that when e is removed from G, there still exists a spanning tree. We know a graph is defined to be connected if given 2 vertices $v, w \in G$, there exists a path between v and w. The spanning tree guarantees that a path exist between any 2 vertices, in turn suggesting that G is connected. However, this is a contradiction since we know that G is disconnected after e is removed. This proves that G no longer has any spanning trees. The only way spanning trees are invalidated is if an edge in the spanning tree is removed. Thus, since all spanning trees of G were invalidated when e was removed, this proves that e must have been a part of every spanning tree in G

7b) If removing e or e' leaves G disconnected, we know that all spanning trees must contain e and e'. Applying this rule to all edges that share the same weight with at least 1 other edge, there are 2 possible outcomes for the remaining edges or unique weight edges that disconnect the graph when removed

① There are no more remaining edges (in other words, no edge has a uniquely distinct weight). In this case, G must already be a tree since every edge is a part of every spanning tree and removing any edge creates a disconnected graph. In this case, G is already the unique minimal spanning tree since if it is a tree itself there is only 1 spanning tree.

② There exist some number of distinct edges that have a unique weight. → 2 more cases: 2.1 $n-1$ edges that have a shared weight = unique MST (as defined by the above problem)

2.2 Proof by contradiction:

Assume T_1 and T_2 are distinct minimal spanning trees that contain every non-unique weighted edge and some of the unique weighted edges. Say there exists a minimum unique weighted edge f_1 that is in T_1 but not T_2. If we were to add f_1 to T_2, this would create a cycle. There must also be an edge f_2 in the cycle that is in T_2 and not in T_1 (or else T_1 would contain a cycle) that has a unique weight. Since f_1 is the minimum unique weight, 7.1 5 / 5

✓ - 0 pts Correct

Proof by contradiction

7a) Assume that when e is removed from G, there
still exists a spanning tree. We know a graph is defined
to be connected if given 2 vertices $v, w \in G$, there exists
a path between v and w. The spanning tree guarantees
that a path exist between any 2 vertices, in turn
suggesting that G is connected. However, this is a contradiction
since we know that G is disconnected after e is removed.
This proves that G no longer has any spanning trees.
The only way spanning trees are invalidated is if an
edge in the spanning tree is removed. Thus, since all
spanning trees of G were invalidated when e was
removed, this proves that e must have been a part
of every spanning tree in G

7b) If removing e or e' leaves G disconnected, we know
that all spanning trees must contain e and e'. Applying
with at least 1 other edge,
this rule to all edges that share the same weight there
or unique weight edges that disconnect the graph when removed
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① There are no more remaining edges (in other words, no
edge has a uniquely distinct weight). In this case, G must
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spanning tree since if it is a tree itself there is only
1 spanning tree.
② There exist some number of distinct edges that have
a unique weight. → 2 more cases: (2.1) $n-1$ edges that have a shared weight = unique MST
(as defined by the above problem)
(2.2) Proof by contradiction:
Assume T_1 and T_2 are distinct minimal spanning trees that contain
every non-unique weighted edge and some of the unique weighted edges.
Say there exists a minimum unique weighted edge f_1 that is in T_1 but not
T_2. If we were to add f_1 to T_2, this would create a cycle. There must
also be an edge f_2 in the cycle that is in T_2 and not in T_1 (or else T_1 would
contain a cycle) that has a unique weight. Since f_1 is the minimum unique weight,

(they cannot be the same since unique)
implying it is shorter than f_2, we know that the length of T_2 can be reduced by adding f_1 and subtracting f_2.

We know the MST remains a MST since removing an edge from a cycle does not change the vertices visited in the MST. However, this is a contradiction since we said T_1 and T_2 are both distinct MST, $\Rightarrow$ the graph G must have a unique minimal spanning tree.

$\rightarrow$ but we have proven T_2 is not

7.2 **5 / 5**

✓ **- 0 pts** Correct

8a) If T is a perfect binary tree, it has to be symmetrical. Thus, the right and left subtree must have an equal number of terminal vertices, each half of $2^h = 2^{h-1}$. For either the left or right subtree to have 2^{h-1} terminal nodes, it must also be perfect. We know this since each subtree must have height $h-1$ (1 less than that of T) and the maximum number of terminal nodes of a tree with height $h-1$ is 2^{h-1}, which only happens when the tree is a perfect binary tree.

8b) T_1 and T_2 are isomorphic since we are able to define a bijection from T_1 to T_2. The bijection is a recursive function that maps the root of the tree to the corresponding root on the other tree $f(R_1)=R_2$ and passes the left subtree of the first tree and the left subtree of the second tree (if both are not null) into the function and the right subtree of the first tree and the right subtree of the second tree (if both are not null) into the function. At the base level, each terminal node will be mapped to a corresponding terminal node in the other tree, ending the recursion because the left and right subtrees both are null.

8.1 3 / 5

✓ **- 2 pts** doesn't justify/ mistake in justifying why left and right subtrees each have 2^{h-1} terminal vertices / height $h-1$

8a) If T is a perfect binary tree, it has to be symmetrical. Thus, the right and left subtree must have an equal number of terminal vertices, each half of $2^h = 2^{h-1}$. For either the left or right subtree to have 2^{h-1} terminal nodes, it must also be perfect. We know this since each subtree must have height $h-1$ (1 less than that of T) and the maximum number of terminal nodes of a tree with height $h-1$ is 2^{h-1}, which only happens when the tree is a perfect binary tree.

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8.2 3 / 5

✓ **- 2 pts** gives an isomorphism or good ideas for an isomorphism, but doesn't prove that it is an isomorphism.

*Node/vertex used interchangeably

9a) Let N_3 represent the number of non-terminal nodes with 3 children (full) and N_0 represent the number of terminal nodes. Since there are n vertices, we know $N_0 + N_3 = n$. Additionally the number of edges in a tree must be $n-1$, and each N_3 node connects to 3 edges (full). Thus $3 \cdot N3 = n-1$. Solving this system of equation gives us $N_0 = 2 \times N3 + 1$. The bijection between the set of nonisomorphic 3-ary trees with n vertices and the set of nonisomorphic full 3 ary trees is then simply mapping each 3-ary tree with n vertices to a full 3-ary tree where the non-terminal nodes of the full tree are isomorphic to the n vertices tree. It is guaranteed that there will be $2n+1$ terminal nodes by $N_0 = 2 \times N_3 + 1$ where n is the number of non-terminal nodes as a result of the mapping.

vertices tree

9b) If there are n nodes, 1 must be set to the root, leaving $n-1$ nodes to be distributed in the left, middle, and right subtrees. If we define i to be the number of nodes in the left tree and j to be the number of nodes in the middle tree, this leaves $n-1-(i+j)$ nodes to be in the right tree. The number of ways we can arrange the nodes in the left, middle, and right subtrees into nonisomorphic 3-ary trees can be defined by t_i, t_j, and $t_{n-1-(i+j)}$ respectively. Thus, we can find the total number of nonisomorphic 3-ary trees with n nodes for a set i, j ($n-1-(i+j)$ just depends on the previous 2 variables) by using multiplication principle $= t_i t_j t_{n-1-(i+j)}$. We can then do a summation of all possible j values (0 up to $n-1-i$) for a set i and sum all of these for all possible i (0 up to $n-1$). This gives us the total number of isomorphic 3-ary trees with n vertices $t_n = \sum_{i=0}^{n-1} \left(\sum_{j=0}^{n-1-i} t_i t_j t_{n-1-(i+j)} \right)$, which finds the number recursively.

9.1 **4 / 5**

✓ **- 1 pts** only does one direction of the bijection/ doesn't argue why it is 1-1 and onto

*Node/vertex used interchangeably

9a) Let N_3 represent the number of non-terminal nodes with 3 (full) children and N_0 represent the number of terminal nodes. Since there are n vertices, we know $N_0 + N_3 = n$. Additionally the number of edges in a tree must be $n-1$, and each N_3 node connects to 3 (full) edges. Thus $3 \cdot N3 = n-1$. Solving this system of equation gives us $N_0 = 2 \times N3 + 1$. The bijection between the set of nonisomorphic 3-ary trees with n vertices and the set of nonisomorphic full 3 ary trees is then simply mapping each 3-ary tree with n vertices to a full 3-ary tree where the non-terminal nodes of the full tree are isomorphic to the n vertices tree. It is guaranteed that there will be $2n+1$ terminal nodes by $N_0 = 2 \times N_3 + 1$ where n is the number of non-terminal nodes as a result of the mapping.

9b) If there are n nodes, 1 must be set to the root, leaving $n-1$ nodes to be distributed in the left, middle, and right subtrees. If we define i to be the number of nodes in the left tree and j to be the number of nodes in the middle tree, this leaves $n-1-(i+j)$ nodes to be in the right tree. The number of ways we can arrange the nodes in the left, middle, and right subtrees into nonisomorphic 3-ary trees can be defined by t_i, t_j, and $t_{n-1-(i+j)}$ respectively. Thus, we can find the total number of nonisomorphic 3-ary trees with n nodes for a set i, j ($n-1-(i+j)$ just depends on the previous 2 variables) by using multiplication principle $= t_i t_j t_{n-1-(i+j)}$. We can then do a summation of all possible j values (0 up to $n-1-i$) for a set i and sum all of these for all possible i (up to $n-1$). This gives us the total number of isomorphic 3-ary trees with n vertices $t_n = \sum_{i=0}^{n-1} \left(\sum_{j=0}^{n-1-i} t_i t_j t_{n-1-(i+j)} \right)$, which finds the number recursively.

9.2 5 / 5

✓ **- 0 pts** Correct