Math 61 Final Exam
Winter 20

You have until 11:59 pm on Wednesday 3/18/20 to upload a scan of the exam to gradescope. Please put each problem on a separate page (problems can take more than 1 page if you like, and subproblems can go on the same page) and make sure everything is very neat.

You are free to use any resources you like such as the notes, the text, and the internet. You may not collaborate with other students or solicit or give help to anyone else. In particular you can't post exam questions on Q&A sites.

If you have any questions please email me.

Make sure to justify all your answers!

Question 1 (5 points). Show that $4^n = \sum_{i=0}^{n} 3^i \binom{n}{i}$.

The binomial theorem tells us that $(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$. Setting $x = 1$ and $y = 3$ gives us that $4^n = \sum_{i=0}^{n} 3^i \binom{n}{i}$.

Question 2 (5 points). A team of 53 football players have uniforms that are numbered 1 through 53. They stand in a line (not necessarily in order of their numbers). Show that there are 5 players in a row whose numbers sum to at least 131.

Set the players off in 10 blocks of 5 and 1 block of three based on the order they are in, so the first 5 players are in one block, the next 5 in another, and the last 3 in block of three. There are 11 of these blocks. Since $\sum_{i=1}^{5} 3i = 1431$ and $1431/11 > 130$, the average sum of the numbers of the players in a block is greater than 130, so by the pigeonhole principle some block has a sum of 131 or more.

It the block that sums to 131 is a block of five we have our five players in a row, and if it is the block consisting of the last three players then adding the two players immediately before them gives us our five players.

Question 3. For the following two statements either prove that they are correct or give a counterexample.

(1) (2.5 points) If G is a weighted graph where the weights of the edges are not all unique then there are two vertices in G with more than 1 shortest path between them.
(2) (2.5 points) If G is a weighted graph where the weights of the edges are all unique then given any two vertices in G there is a unique shortest path between them.

Neither statement is true. For a counterexample to statment one, take a tree with 3 vertices and both edges of weight one. For statement two take the 2-cube with the weight of the bottom edge 2, the right edge 3, left edge 1, and top edge 4. Then there are two shortest length paths from the lower left hand vertex to the upper right hand vertex.

Question 4. (1) (5 points) Show that simple graphs G_1, G_2 are isomorphic if and only if their complements[1] $\bar{G_1}$ and $\bar{G_2}$ are isomorphic.

(2) (2 points) Show that a simple graph G with n vertices is r-regular[2] if and only if its complement $\bar{G}$ is $n-r-1$ regular.

(3) (2 points) Show that if a graph G is 1-regular then G has an even number of vertices.

(4) (4 points) For n an even number determine the number of nonisomorphic of simple 1-regular graphs with n-vertices.

(5) (2 point) For n an even number determine the number of nonisomorphic simple $n-2$-regular graphs with n-vertices.

(1) First, if G_1 being isomorphic to G_2 implies that $\bar{G_1}$ and $\bar{G_2}$ are isomorphic, then because $\bar{\bar{G}}$ we have that if $\bar{G_1}$ and $\bar{G_2}$ are isomorphic so are $\bar{\bar{G_1}}$ and $\bar{\bar{G_2}}$. So, we only need to prove one direction of the statement.

Suppose that G_1 and G_2 are isomorphic. So, there is a bijection $f : V_1 \to V_2$, where V_i is the set of vertices of G_i, with the property that $x, y \in V_1$ are adjacent if and only if $f(x)$ and $f(y)$ are adjacent.

Since each V_i is also the set of vertices of $\bar{G_i}$, f also gives a bijection between the vertices of $\bar{G_1}$ and the vertices of $\bar{G_2}$. Two vertices $x, y \in V_1$ are adjacent in $\bar{V_1}$ if and only if they are not adjacent in G_1, which happens if and only if $f(x)$ and $f(y)$ are not adjacent in G_2, which happens if and only if $f(x)$ and $f(y)$ are adjacent in $\bar{G_2}$.

We conclude that f is an isomorphism from $\bar{G_1}$ to $\bar{G_2}$.

(2) If a simple graph G with n is r-regular, then that means every vertex of G is adjacent to r vertices among the $n-1$ vertices it could be adjacent to. So, every vertex of $\bar{G}$ is adjacent to the other $n-1-r$ vertices.

Again, because $\bar{\bar{G}} = G$ we only need to prove one direction of the statement.

(3) If the graph is 1-regular then the sum of the degrees of the graph is equal to the number of vertices in the graph. The sum of the degrees is an even number, so there must be an even number of vertices.

(4) Up to isomorphism there is only 1 r-regular graph with n-vertices.

First, note that if a graph is 1-regular it is necessarily simple. Given two 1-regular graphs with n-vertices G_1, G_2 partition each of the vertex sets into sets of size two, where the two vertices in each set are adjacent. Each of these partitions has $n/2$ blocks, and we get an isomorphism $f : G_1 \to G_2$ by matching up the blocks in the two partitions and choosing a bijection between the pairs of vertices in the corresponding blocks.

[1] The complement of a simple graph is defined right before exercise 8.6.36 on page 422 of your text.

[2] r-regular graphs are defined right before exercise 8.6.17 on page 421 of your text.

(5) Part 1 and part 2 applied to part 3 show that up to isomorphism there is only 1 simple $n-2$ regular graph if n is even.

Question 5. In this problem you will model an infectious disease.

Here is the model for the disease:

On day zero and all the days before that 0 people have the disease. On day 1, 1 person has the disease. After 2 days of having the disease everyone who is infected with the disease infects 2 new people perday. So, if someone is infected on day 15 they'll start infecting people on day 17. People that are infected stay infected forever.

(1) (5 points) Let s_n be the number of infected people on the n^{th} day according to the model. Find a recurrence relation for s_n. Give the recurrence relation and initial conditions necessary to determine the sequence. Be sure to justify your answer.
(2) (2 points) Solve this recurrence to find a formuala for s_n.
(3) (3 points) After more observation the model changes: after 10 days of being sick infected people recover from the disease and are no longer affected. On the 11^{th} day of infection you still infect two more people before recovering. Write a new recurrence relation for s_n that incorporates this information and give the initial conditions necessary to determine the sequence. Be sure to justify your answer.
(4) (5 points) Does the number of infected people (according to the model from the 3rd part of this question) keep getting bigger every day after the second day or does it eventually decline (or perhaps sometimes increase and sometimes decrease?). You'll receive full points for this problem if you solve it with techniques we talked about in class rather than solving the recurrence relation.

(1) $s_n = s_{n-1} + 2s_{n-2}$. The s_{n-1} factor counts all the people who were sick on the previous day and the $2s_{n-2}$ factor counts the people who are newly sick on the n^{th} day (they were all infected by someone who was sick on day $n-2$). The initital conditions are $s_0 = 0$ and $s_1 = 1$.
(2) The characteristic polynomial is $x^2 - x - 2 = (x-2)(x+1)$. So, the solution is $s_n = a2^n + b(-1)^n$ and we need to determine a and b. Since $s_0 = 0$ we have that $b = -a$. So, $1 = s_1 = a2 - a(-1)$, and $a = 1/3$ and $s_n = (1/3)2^n - (1/3)(-1)^n$.
(3) If after 10 days people get better we need to subtract the number of people that got sick on the day $n-10$ from s_{n-1}. If we let t_n be the number of people that got sick on day n we have that $s_n = s_{n-1} - t_{n-10} + 2s_{n-1}$, and $s_n = \sum i = 0^9 t_{n-i}$, so $s_n - s_{n-1} = t_n - t_{n-10}$. This gives is the equations $s_n - s_{n-1} = 2s_{n-1} - t_{n-10}$ and $s_n - s_{n-1} = t_n - t_{n-10}$, so we conclude that $t_n = 2s_{n-2}$.

   This gives the recurrence $s_n = s_{n-1} - 2s_{n-12} + 2s_{n-10}$.
(4) The sequence is increasing. We show by strong induction that for $j > 2$ that $i < j$ implies that $s_i < s_j$.

   The base base is s_3 through s_{15}. You could just compute this directly. Or, you could show that s_3 through s_12 is increasing since you aren't subtracting anything, and then check s_{13} through s_{15} by noting that you're never subtracting more than s_5 and then noting that $2s_5 = 22$ and $s_7 = 43$.

Our inducive hypothesis is that for some $k > 15$ we have that for all $2 < j < k$ that if $i < j$ then $s_i < s_j$. We consider $s_k = s_{k-1} - 2s_{k-12} + 2s_{k-2}$. We want to show that $s_k - s_{k-1} > 0$. We see that $s_k - s_{k-1} = 2(s_{k-2} - s_{k-12})$. But our inductive hypothesis tells us that $s_{k-2} > s_{k-12}$ so this quantity is positive and we're done by induction.

Question 6 (5 points)*.* In a planar embedding of a connected planar graph with a cycle every face is bounded by a cycle.

Show that every planar embedding of a connected planar graph with a cycle has an even number of faces that are bounded by cycles of odd length.

Note: This question is wrong as stated. Instead of "cycle" it should say "closed path" because the boundary of a face isn't necessarily a cycle because there could be repeated edges.

Suppose that G is a connected planar graph. Take the subgraph G' consisting of all the vertices and edges that are the boundary of a face in G. Note that G' has the same faces as G and each face has the same boundary. Consider the set F of faces of G' and consider the sum

$$\sum_{f \in F} \#\text{ of edges in the boundary of } f.$$

Since each edge in G' is part of the boundary exactly two faces or it appears twice in the boundary of one face this sum is twice the number of edges in G' and is an even number. So, there must be an even number of odd numbers appearing in this sum, so there is an even number of faces whose boundary has odd length.

You don't need to consider this subgraph G' if you don't want to. Another solution would be to make a new graph D by having the vertices of D be the faces of G with an edge between two vertices for each edge in G that the faces share in common. Then the result follows from the handshaking lemma.

Question 7. You have 6 coins and exactly one coin is lighter or heavier than the others.

You have access to a balance scale that can determine whether or not one object is heavier than another one or whether the objects are the same weight.

(1) (5 points) Describe an algorithm to find the odd coin and determine whether it is lighter or heavier. Also describe why your algorithm is correct.
(2) (5 points) Prove that your solution is the best possible, i.e. there is no algorithm that can solve the problem in fewer weighings than your algorithm.

(1) Call the coins A, B, C, D, E, F. I'll describe the decision tree. Start by weighing AB vs CD.

    Suppose first that the left hand side is heavier. Now weigh $ACvsBD$. If the left hand side is still heavier the odd coin is A or D and A is heavy or D is light. If the right hand side is heavier the odd coin is B or C and B is heavy or C is light. If the odd coins is A or D weigh A again E. Since E is normal if A is heavier then A is the odd coin and it's heavier. If A and E weight the same we conclude that D is the odd coin and it is light. Similarly, if B is heavy or C is light we can weight B against E to determine which is which. This takes a total of three weighings.

    If CD is heavier than AB then interchange the symbols A and D and B and C and run the algorithm in the second paragraph (the situation is symmetric).

    If AB is the same weight as CD the odd coin is E or F. We can weigh A against E and if necessary A against F to determine the odd coin and whether it is heavier or light.

(2) Since there are 6 coins there are a total of 12 possibilities. So, any decision tree must have at leave 12 terminal vertices, so its height must be greater than $\log_3 12$, so height 3 or greater.

Question 8 (5 points). Let G be a weighted simple graph with one edge e whose weight is less than the weight of any other edge in G. On your homework you showed that e is an edge in in every minimal spanning tree of G.

Show that if there is an edge e' whose weight is less than that of every other edge in G other than e, then e' is also an edge in every minimal spanning tree of G.

Suppose for a contradiction that G has a MST T that doesn't include e'. Adding e' to T gives a graph with a cycle involving e'. Since the graph is simple any cycle has lenght 3 or longer, so it involves and edge e'' that isn't e or e''. Removing this edge e'' gives a spanning tree of weight less than T because the weight of e' is less than that of e'', which contradicts that T was a MST.

Question 9 (5 points). Let X be the set $\{1, \ldots, n\}$. Construct a bijection from the set of symmetric and reflexive relations on X to the set of subgraphs of K_n that have exactly n-vertices.

First, we can label K_n by the elements of X.

Given a subgraph H of K_n with n-vertices define an relation $R(H)$ on X by $(x, y) \in R(H)$ if x and y are adjacent in H, and for all $x \in X$, $(x, x) \in R(H)$. The relation $R(H)$ is symmetric since if x and y are adjacent then y and x are adjacent, and it is symmetric by construction. This gives us a function $R :$ subgraphs of K_n with n vertices $\to$ symmetric and reflexive relation on X.

Let's define the inverse function. Given a reflexive relation S on X define a subgraph graph $G(S)$ of K_n containg all the vertices of K_n by for $x \neq y \in X$ we have that x and y are adjacent in $G(S)$ if $(x, y) \in S$. This gives a function $G :$ symmetric and reflexive relation on $X \to$ subgraphs of K_n with n vertices.

We can check that $G \circ R = id$ and $R \circ G = id$, if we take a graph H then x, y with $x \neq y$ are adjacent in $G \circ R(H)$ if and only if $(x, y) \in R(H)$, which occurs if and only if x and y are adjacent in H, so $H = G \circ R(H)$ for any H a subgraph of K_n with n vertices. Similarly if we take a relation S then $(x, y) \in R \circ G(S)$ if and only if either $x = y$ or if x, y are adjacent in $G(S)$, which occurs if and only if $x \neq y$ and $(x, y) \in S$. So $R \circ G(S) = S$ for any relation S on X and we conlude that G and R are each bijections.

Question 10. In this question you'll show that the Catalan numbers satisfy the recurrence $C_n = \frac{2(2n-1)}{n+1}C_{n-1}$. This is basically problems 27-32 of section 9.8 of your text.

Set X_1 to be the number of nonisomorphic full binary trees with n terminal vertices, and X_2 to be the number of nonisomorphic full binary trees with $n+1$ terminal vertices where one of the terminal vertices is marked.

Given a tree T in X_1 and a vertex v in T, construct two trees in X_2 as follows:

We either replace v with a vertex v' and make the left subtree of v' the subtree of T rooted at v and we make the right child of v' a marked terminal vertex, *or* we replace v with a vertex v' and make the *right* subtree of v' the subtree of T rooted at v and make the *left* child of v' a marked terminal vertex.

See figures $1-3$ for any example of this process of making trees in X_2 from a tree T in X_1 and a vertex in T.

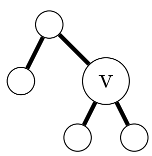

FIGURE 1. The tree and the vertex v that we are going to use to make by two trees in X_2

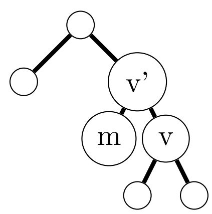

FIGURE 2. Adding a marked left child labeled m

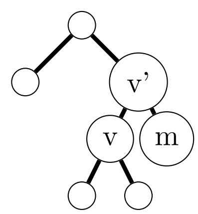

FIGURE 3. Adding a marked right child labeled m

Let X_T denote the set of all such trees constructed from T (as v ranges over all the vertices of T).

(1) (5 points) Construct a bijection from the set of nonisomorphic binary trees with $n-1$ vertices where $n \geq 1$ to X_1 by adding children to all the vertices of a binary tree with $n-1$ vertices that don't already have two children. You neeed to show that this is a function with the claimed domain and codomain and that it is one to one and onto. Conclude that $|X_1| = C_{n-1}$ and that $|X_2| = (n+1)C_n$.
(2) (7 points) Show that $|X_T| = 2(2n-1)$ for every $T \in X_1$.
(3) (6 points) Show that $\{X_T : T \in X_1\}$ is a partition of X_2.
(4) (2 points) Show that $|X_2| = 2(2n-1)|X_1|$ and that $C_n = \frac{2(2n-1)}{n+1}C_{n-1}$.

(1) That this is a function is clear enough and it's also clear than the resulting graph is a full binary tree, what is not immediately clear is that the resulting graph is in X_1, i.e. that if we do this procedure to a binary tree with $n-1$ vertices we'll get a full binary tree with n vertices. But note that every vertex in the old graph becomes an internal vertex in the new graph, so since the new graph has $n-1$ internal vertices it must have n terminal vertices.

    To see that this is a bijection contruct the inverse function: given a full binary tree with n terminal vertices get a binary tree with $n-1$ vertices by deleting all of the terminal vertices. The argument from the previous paragraph shows that this has the right domain and codomain and the functions are inverse to each other. Since the number of binary trees is with $n-1$ vertices is equal to C_{n-1} we have that $|X_1| = C_{n-1}$. Since a tree in X_2 has $n+1$ terminal vertices and any one of them could be marked we get that $|X_2| = (n+1)C_n$.

(2) For this, we just need to show that given a tree in T we get 2 distinct graphs for each of the $2n-1$ vertices we choose in G and that there are no overlaps. However, given a graph in X_T we can recover determine which vertex was chosen an whether or not we chose to add a marked left vertex or a marked right vertex: if we have a graph T in X_T, the parent of the marked vertex is the chosen vertex. If it's a left child then we added a marked left vertex, and vice versa if it's a right child.

    We recover T be deleting the marked vertex and moving the subtree rooted at the sibling of the marked vertex to the spot where the marked vertex parent is.

(3) For this we need to show that for $T_1 \neq T_2$ that $X_{T_1} \cap X_{T_2} = \varnothing$ and that $\cup_{T\in X_1} X_T = T_2$. The first claim follows from the previous item, it tells us in particular that given a tree in X_2 we can determine which tree in X_1 was operated on to give the tree in X_2 (Given $G \in X_T$ we can recover T by deleting the marked vertex and replacing the parent of the marked vertex with the subtree rooted at the sibling of the marked vertex).

    The previous item also tells us that second claim, given some $T \in X_2$ we have a procedure to make a tree T' so that $T \in X_{T'}$. I repeat myself by saying that we determine T' by deleting the marked vertex in T and moving the subtree rooted at the sibling of the marked vertex in T to the spot where the marked vertex parent is.

(4) Since the $\{X_T : T \in X_1\}$ partition X_2 then $|X_2| = \sum_{T\in X_1} |X_T|$. Each of these has size $2(2n-1)$, so we get that $|X_2| = 2(2n-1)|X_1|$, no the second part about the Catalan numbers follows from part one.

    Another way of phrasing this is to define a function $f : X_2 \to X_1$ by $f(G)$ is the graph obtained by deleting the marked vertex of G and its parent and placing the parent with the subtree rooted at the sibling of the marked vertex. Then $f^{-1}(T) = X_T$. The function f is surjective because for any $G \in X_T$ $f(G) = T$. So, $\{f^{-1}(T) : T \in X_1\}$ is a partition of X_2 by the last problem on the first midterm.