Math 61 Final Exam
Winter 20

You have until 11:00 pm on Wednesday 3/18/20 to upload a scan of the exam to gradescope. Please put each problem on a separate page (problems can take more than 1 page if you like, and subproblems can go on the same page) and make sure everything is very neat.

You are free to use any resources you like such as the notes, the text, and the internet. You may not collaborate with other students or solicit or give help to anyone else. In particular you can't post exam questions on Q&A sites.

If you have any questions please email me.

Make sure to justify all your answers!

Question 1 (5 points). Show that $4^n = \sum_{i=0}^{n} 3^i \binom{n}{i}$.

Question 2 (5 points). A team of 53 football players have uniforms that are numbered 1 through 53. They stand in a line (not necessarily in order of their numbers). Show that there are 5 players in a row whose numbers sum to at least 131.

Question 3. For the following two statements either prove that they are correct or give a counterexample.

(1) (2.5 points) If G is a weighted graph where the weights of the edges are not all unique then there are two vertices in G with more than 1 shortest path between them.
(2) (2.5 points) If G is a weighted graph where the weights of the edges are all unique then given any two vertices in G there is a unique shortest path between them.

Question 4.
(1) (5 points) Show that simple graphs G_1, G_2 are isomorphic if and only if their complements[1] $\bar{G}_1$ and $\bar{G}_2$ are isomorphic.
(2) (2 points) Show that a simple graph G with n vertices is r-regular[2] if and only if its complement $\bar{G}$ is $n - r - 1$ regular.
(3) (2 points) Show that if a graph G is 1-regular then G has an even number of vertices.
(4) (4 points) For n an even number determine the number of nonisomorphic 1-regular graphs with n vertices.
(5) (2 points) For n an even number determine the number of nonisomorphic simple $n-2$-regular graphs with n vertices.

Question 5. In this problem you will model an infectious disease.

Here is the model for the disease:

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[1]The complement of a simple graph is defined right before exercise 8.6.36 on page 422 of your text.

[2]r-regular graphs are defined right before exercise 8.6.17 on page 421 of your text.

On day zero and all the days before that 0 people have the disease. On day 1, 1 person has the disease. After 2 days of having the disease everyone who is infected with the disease infects 2 new people per day. So, if someone is infected on day 15 they'll start infecting people on day 17. People that are infected stay infected forever.

(1) (5 points) Let s_n be the total number of infected people on the n^{th} day according to the model (i.e. all the people that are infected, not just those who are infected on the n^{th} day). Find a recurrence relation for s_n. Give the recurrence relation and initial conditions necessary to determine the sequence. Be sure to justify your answer.
(2) (2 points) Solve this recurrence to find a formuala for s_n.
(3) (3 points) After more observation the model changes: after 10 days of being sick infected people recover from the disease and are no longer affected. On the 10^{th} day of infection you still infect two more people before recovering. Write a new recurrence relation for s_n that incorporates this information and give the initial conditions necessary to determine the sequence. Be sure to justify your answer.
(4) (5 points) Does the number of infected people (according to the model from the 3rd part of this question) keep getting bigger every day or does it eventually decline (or perhaps sometimes increase and sometimes decrease?). You'll receive full points for this problem if you solve it with techniques we talked about in class rather than solving the recurrence relation.

Question 6 (5 points). In a planar embedding of a connected planar graph with a cycle every face is bounded by a cycle.

Show that every planar embedding of a connected planar graph with a cycle has an even number of faces that are bounded by cycles of odd length.

Question 7. You have 6 coins and exactly one coin is lighter or heavier than the others.

You have access to a balance scale that can determine whether or not one object is heavier than another one or whether the objects are the same weight.

(1) (5 points) Describe an algorithm to find the odd coin and determine whether it is lighter or heavier.
(2) (5 points) Prove that your solution is the best possible, i.e. there is no algorithm that can solve the problem in fewer weighings than your algorithm.

Question 8 (5 points). Let G be a weighted simple graph with one edge e whose weight is less than the weight of any other edge in G. On your homework you showed that e is an edge in in every minimal spanning tree of G.

Show that if there is an edge e' whose weight is less than that of every other edge in G other than e, then e' is also an edge in every minimal spanning tree of G.

Question 9 (5 points). Let X be a set with n elements. Construct a bijection from the set of symmetric and reflexive relations on X to the set of subgraphs of K_n that have exactly n vertices.

Question 10. In this question you'll show that the Catalan numbers satisfy the recurrence $C_n = \frac{2(2n-1)}{n+1} C_{n-1}$. This is basically problems 27-32 of section 9.8 of your text.

Set X_1 to be the set of nonisomorphic full binary trees with n terminal vertices, and X_2 to be the set of nonisomorphic full binary trees with $n+1$ terminal vertices where one of the terminal vertices is marked.

Given a tree T in X_1 and a vertex v in T, construct two trees in X_2 as follows:

We either replace v with a vertex v' and make the left subtree of v' the subtree of T rooted at v and we make the right child of v' a marked terminal vertex, *or* we replace v with a vertex v' and make the *right* subtree of v' the subtree of T rooted at v and make the *left* child of v' a marked terminal vertex.

See figures $1-3$ for any example of this process of making trees in X_2 from a tree T in X_1 and a vertex in T.

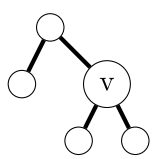


FIGURE 1. The tree and the vertex v that we are going to use to make two trees in X_2

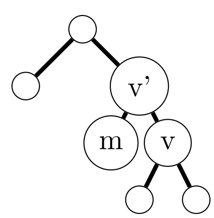


FIGURE 2. Adding a marked left child labeled m

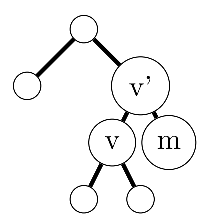


FIGURE 3. Adding a marked right child labeled m

Let X_T denote the set of all such trees constructed from T (as v ranges over all the vertices of T).

(1) (5 points) Construct a bijection from the set of nonisomorphic binary trees with $n-1$ vertices to X_1 by adding children to all the vertices of a binary tree with $n-1$ vertices that don't already have two

children. You neeed to show that this is a function with the claimed domain and codomain and that it is one to one and onto. Conclude that $|X_1| = C_{n-1}$ and that $|X_2| = (n+1)C_n$.

(2) (5 points) Show that $|X_T| = 2(2n-1)$ for every $T \in X_1$.
(3) (8 points) Show that $\{X_T : T \in X_1\}$ is a partition of X_2.
(4) (2 points) Show that $|X_2| = 2(2n-1)|X_1|$ and that $C_n = \frac{2(2n-1)}{n+1} C_{n-1}$.

20W-MATH61-2 Final Exam

NEIL VAISHAMPAYAN

TOTAL POINTS

68.5 / 90

QUESTION 1

1 Question One 5 / 5

- ✓ + **5 pts** Correct
- \+ **3 pts** (Partial) Stated binomial theorem
- \+ **2 pts** (Partial) Correct set up an induction

QUESTION 2

2 Question Two 4 / 5

- ✓ + **1 pts** Tried pigeonhole
- ✓ + **1 pts** A correct application of pigeonhole given the context of the problem (may not be the relevant one towards the solution, but as long as pigeons and holes are explained and the conclusion deduced correctly from that, the points are obtained (or do the argument with the average of numbers, etc))
- \+ **1 pts** Explained how the 11th group of 3 creates a group of 5, or how to do the overlapping group of 5
- ✓ + **2 pts** A pigeonhole application that actually leads to a solution of the problem (can only get pts here if you actually do it correctly)
- \- **1 pts** Other mistake that was commented on
- \+ **0 pts** No points

1 How do you deal with the 11th group of 5?

QUESTION 3

Question Three 5 pts

3.1 Part 1 2.5 / 2.5

- ✓ + **2.5 pts** Correct
- \+ **1 pts** Tried proof, said some correct stuff for general graphs
- \+ **0 pts** No points
- \+ **1 pts** Slightly incorrect counterexample

3.2 Part 2 2.5 / 2.5

- ✓ + **2.5 pts** Correct
- \+ **1 pts** Tried proof, said some correct stuff for general graphs
- \+ **0 pts** No points
- \+ **1 pts** Slightly incorrect counterexample

QUESTION 4

Question Four 15 pts

4.1 Part 1 3 / 5

- ✓ - **2 pts** Used something we didn't talk about without justifying why it's true

4.2 Part 2 1.5 / 2

- \+ **2 pts** Correct
- ✓ + **1.5 pts** only does 1 direction
- \+ **1 pts** some misunderstanding of complements
- \+ **1 pts** uses stuff we didn't talk about in class without justifying it
- \+ **0 pts** blank/ no progress towards solution

You say iff at the end but your logic only goes in 1 direction

4.3 Part 3 2 / 2

- ✓ + **2 pts** Correct
- \+ **1 pts** issue in logic
- \+ **0 pts** blank/ no progress towards solution

4.4 Part 4 4 / 4

- ✓ + **4 pts** Correct
- \+ **1 pts** not answering this question
- \+ **2 pts** uses stuff we didn't talk about
- \+ **3 pts** missing detail
- \+ **0 pts** blank/ no progress towards solution.
- \+ **3 pts** some mistake in understanding what

"isomorphic means"

+ **0 pts** Click here to replace this description.

4.5 Part 5 **2 / 2**

✓ + **2 pts** **Correct/ correct based on answer to previous question**

+ **0 pts** blank/ no progress towards the solution

+ **1 pts** some misunderstanding about isomorphisms

+ **1 pts** missing justification/ problems with justification

QUESTION 5

Question 5 15 pts

5.1 Part 1 **5 / 5**

✓ + **3 pts** **Having idea of cases to figure out people sick on nth day**

✓ + **1 pts** **Fully correct relation**

✓ + **1 pts** **Initial conditions**

5.2 Part 2 **2 / 2**

✓ + **2 pts** **Correct, based on 1**

+ **1 pts** Small mistake, based on 1

+ **0 pts** No points

5.3 Part 3 **1 / 3**

✓ + **0.5 pts** **Idea of subtracting sick people from 10 days ago**

✓ + **0.5 pts** **Correct indexing of n - 10 (or n - 12 if correct)**

+ **0.5 pts** Knew that - s_{n - 10} was not correct, tried to subtract only people who got sick on the 10th previous day

+ **0.5 pts** Fully correct

+ **1 pts** Give enough initial conditions

+ **0 pts** No points

5.4 Part 4 **0 / 5**

✓ + **0 pts** **No points**

+ **3 pts** Idea to use (strong) induction

+ **1 pts** Handled base cases correctly, if using induction

+ **1 pts** Inductive step correct

+ **2 pts** Daily analysis of people recovering always infecting 2 more people

+ **3 pts** If doing a daily analysis, then noting that (besides base cases) 10 days ago there were people infected OR doing a correct dichotomy

QUESTION 6

6 Question 6 **2 / 5**

+ **2.5 pts** Every edge is on the boundary of 2 faces or 0 faces

+ **2.5 pts** Therefore sum of face boundary lengths must be even

✓ + **2 pts** **(partial) Argument has some correct statements but contains a few errors**

+ **1 pts** (partial) Argument has some correct statements but contains errors.

+ **0 pts** Blank or incorrect

(2) This process can affect the faces with odd boundary length too. What if you have an edge that is a part of 2 cycles, one of odd length and one of even length?

QUESTION 7

Question 7 10 pts

7.1 Part 1 **3.5 / 5**

✓ + **3.5 pts** **Algorithm correctly finds the odd coin and its weight**

+ **1.5 pts** Correct explanation why algorithm works

+ **0 pts** Blank or incorrect

+ **2 pts** (partial) Algorithm finds the odd coin but not whether it is lighter or heavier

+ **1 pts** (partial) Algorithm distinguishes between most cases

7.2 Part 2 **5 / 5**

✓ + **2 pts** **There are 12 possible cases**

✓ + **1 pts** **Algorithm is modeled by $$3$$\-ary decision tree**

✓ + **2 pts** **Tree of height $$2$$ can only have $$9$$**

leaves

- 2 pts Algorithm from (a) takes more than $$3$$ weighings

+ 0 pts Blank or incorrect

QUESTION 8

8 Question 8 2.5 / 5

+ 1 pts Consider an MST that doesn't contain $$e'$$

+ 2 pts Adding $$e'$$ to this MST must create a cycle

+ 1 pts This new cycle has length at least 3

+ 1 pts So there is an edge in this cycle of weight more than that of $$e'$$, a contradiction

✓ + 2.5 pts **(partial) Different argument that makes some progress (might include some of the items from above, but fewer then 2.5 points worth)**

+ 1 pts (partial) Different argument makes a little progress

+ 0 pts Blank or incorrect

3 This is a good argument that any MST made through Prim's algorithm will contain e'. But how do we know that every MST can be constructed in this way?

QUESTION 9

9 Question 9 5 / 5

✓ + 1 pts **Function has correct domain and codomain.**

✓ + 2 pts **Correct idea for the function**

✓ + 2 pts **Argued that function is invertible, or that function is injective and surjective**

+ 1.5 pts (partial) Said size of set was $$2^{\binom{n}{2}} = 2^{\frac{n(n-1)}{2}}$$

+ 0 pts Blank or incorrect

QUESTION 10

Question 10 20 pts

10.1 Part 1 5 / 5

✓ - 0 pts **correct**

10.2 Part 2 3 / 5

✓ - 2 pts **Big mistake or missing that these different graphs are all unique.**

10.3 Part 3 6 / 8

✓ - 2 pts **smaller mistake showing that they are disjoint**

10.4 Part 4 2 / 2

✓ + 2 pts **Correct**

+ 1 pts some problem with using the partition

+ 0 pts blank

Neil Vaidhampayan

Math 61 Final Exam

1) Consider a string of length n formed with only the characters 0, 1, 2, 3.

Method 1: Since there are 4 possible options for every character in the string, we find that through the multiplication principle there are $\underline{4^n}$ strings of length n.

Method 2: Another way to form the strings of length n made out of the numbers 0, 1, 2, 3 is to sum all the strings formed from 0, 1, 2, 3 that contain i nonzero characters. These strings will have $n-i$ 0s, and the amount of positions these 0s can take is $\binom{n}{n-i}$ which is equivalent to $\binom{n}{i}$. Then since there are 3 choices for each of the rest of the characters, and there are i nonzero characters, there are 3^i combinations of the nonzero numbers. Thus, the amount of strings of length n with i nonzero numbers is $3^i\binom{n}{i}$.

So the total amount of strings of length n formed from 0, 1, 2, 3 is given by summing the amount of strings with i nonzero characters from $0 \le i \le n$, or $\underline{\sum_{i=0}^{n} 3^i\binom{n}{i}}$.

These expressions are equivalent, thus $\boxed{4^n = \sum_{i=0}^{n} 3^i\binom{n}{i}}$

1 Question One **5 / 5**

- ✓ + **5 pts** **Correct**
- + **3 pts** (Partial) Stated binomial theorem
- + **2 pts** (Partial) Correct set up an induction

2) We can use the pigeonhole principle to demonstrate this statement.

Suppose that each group of 5 consecutive players is a hole. In a line of 53 players, there are 11 holes.

Now let there be pigeons equal to the sum of the shirt numbers from 1 to 53. So there are 1431 pigeons.

According to the pigeonhole principle, there must be at least one hole that contains $\lceil \frac{1431}{11} \rceil$ or 131 pigeons.

Thus the sum of one group of 5 consecutive players must be at least 131.

2 Question Two **4 / 5**

✓ **+ 1 pts** **Tried pigeonhole**

✓ **+ 1 pts** **A correct application of pigeonhole given the context of the problem (may not be the relevant one towards the solution, but as long as pigeons and holes are explained and the conclusion deduced correctly from that, the points are obtained (or do the argument with the average of numbers, etc))**

+ 1 pts Explained how the 11th group of 3 creates a group of 5, or how to do the overlapping group of 5

✓ **+ 2 pts** **A pigeonhole application that actually leads to a solution of the problem (can only get pts here if you actually do it correctly)**

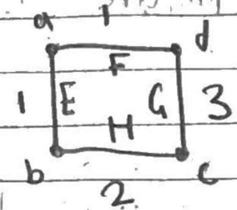
- 1 pts Other mistake that was commented on

+ 0 pts No points

1 How do you deal with the 11th group of 5?

3) (1)

Counterexample:



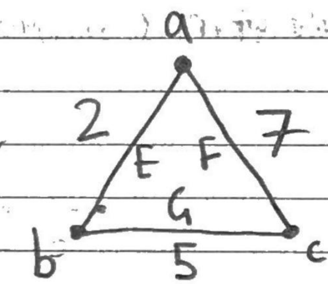
Shortest path between vertices:

(a,b) -> E
(a,c) -> EH
(a,d) -> F
(b,c) -> H
(b,d) -> EF
(c,d) -> G

All vertices have a unique shortest path despite the weight of E being the same as the weight of F.

(2)

Counterexample:



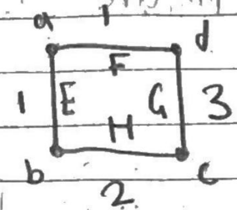
There are 2 shortest paths from a to c, either EG, or F.

3.1 Part 1 **2.5 / 2.5**

- ✓ **+ 2.5 pts** **Correct**
- **+ 1 pts** Tried proof, said some correct stuff for general graphs
- **+ 0 pts** No points
- **+ 1 pts** Slightly incorrect counterexample

3) (1)

Counterexample:



Shortest path between vertices:

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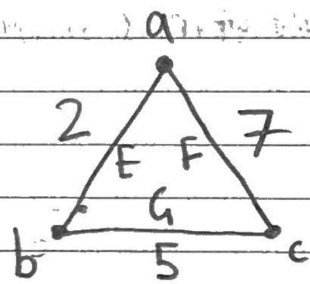
Counterexample:



There are 2 shortest paths from a to c, either EG, or F.

3.2 Part 2 **2.5 / 2.5**

- ✓ **+ 2.5 pts** **Correct**
- **+ 1 pts** Tried proof, said some correct stuff for general graphs
- **+ 0 pts** No points
- **+ 1 pts** Slightly incorrect counterexample

4) (1) The definition of an isomorphic graph is that G_1 and G_2 are isomorphic if and only if, for some ordering of their vertices, their adjacency matrices are equal.

The complement of a graph has 2 vertices adjacent if and only if they were not adjacent in the original graph. Since a simple unweighted graph's adjacency matrix contains a 1 if two vertices are adjacent, and 0 if they are not, and the complement of a simple unweighted graph is also simple and unweighted, the adjacency matrix of the complement simply has 0s where the original matrix has 1, and vice versa. Let f be the transformation that the adjacency matrix undergoes when the adjacency matrix of the graph changes to the adjacency matrix of the complement. It is easily seen that f is a bijection.

Let A be the adjacency matrix of G_1 and B be the adjacency matrix of G_2. Thus, f(A) is the adjacency matrix of $\bar{G}_1$ and f(B) is the adjacency matrix of $\bar{G}_2$. However, since G_1 and G_2 are isomorphic A=B, and f(A)=f(B), thus $\bar{G}_1$ and $\bar{G}_2$ are isomorphic.

Therefore, it follows that G_1 and G_2 are isomorphic if and only if $\bar{G}_1$ and $\bar{G}_2$ are isomorphic.

(2) If a graph G with n vertices is r-regular, every vertex in G has degree r. In $\bar{G}$, two vertices are adjacent if they were not adjacent in G. Since G is a simple graph, if it was connected to r vertices, and there are n-1 possible vertices to be connected with (since a vertex cannot be adjacent to itself), it is not connected to n-r-1 vertices. Thus in the complement, each vertex will be connected to n-r-1 vertices, making $\bar{G}$ n-r-1-regular. Thus, G is r-regular if and only if $\bar{G}$ is n-r-1-regular.

4.1 Part 1 **3 / 5**

✓ **- 2 pts** Used something we didn't talk about without justifying why it's true

4) (1) The definition of an isomorphic graph is that G_1 and G_2 are isomorphic if and only if, for some ordering of their vertices, their adjacency matrices are equal.

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4.2 Part 2 **1.5 / 2**

+ **2 pts** Correct
✓ + **1.5 pts** **only does 1 direction**
+ **1 pts** some misunderstanding of complements
+ **1 pts** uses stuff we didn't talk about in class without justifying it
+ **0 pts** blank/ no progress towards solution

You say iff at the end but your logic only goes in 1 direction

(3) If a graph G is 1-regular with n vertices, since $\sum \deg(v) = 2E$, there are $\frac{n}{2}$ edges. However, there must be an integer amount of edges, and thus n must be even.

(4) All 1-regular graphs with n-vertices will have $\frac{n}{2}$ edges each connected to two vertices. Thus any 1-regular graph can be renamed to match any other 1-regular graph, as there will always be $\frac{n}{2}$ disconnected components of 2 adjacent vertices. Thus there is $\boxed{1}$ distinct (up to isomorphism) 1-regular graph for all even n.

(5) Using the results from parts (1) and (2), since G_1 and G_2 are isomorphic if and only if $\bar{G}_1$ and $\bar{G}_2$ are isomorphic, and G is r-regular if and only if $\bar{G}$ is $n-r-1$ regular, r-regular graphs G_1 and G_2 are only isomorphic if and only if $n-r-1$ regular graphs $\bar{G}_1$ and $\bar{G}_2$ are isomorphic. Thus we can see that the amount of distinct (up to isomorphism) r-regular graphs is equal to the amount of distinct (up to isomorphism) $n-r-1$ regular graphs. Thus, the amount of $n-2$ regular graphs with n vertices is equal to the amount of $n-n+2-1$ regular (or 1-regular) graphs. Using the result of part (4), there is $\boxed{1}$ distinct (up to isomorphism) $n-2$ regular graphs with n vertices, when n is even.

4.3 Part 3 **2 / 2**

- ✓ **+ 2 pts** **Correct**
- **+ 1 pts** issue in logic
- **+ 0 pts** blank/ no progress towards solution

(3) If a graph G is 1-regular with n vertices, since $\sum \deg(v) = 2E$, there are $\frac{n}{2}$ edges. However, there must be an integer amount of edges, and thus n must be even.

(4) All 1-regular graphs with n-vertices will have $\frac{n}{2}$ edges each connected to two vertices. Thus any 1-regular graph can be renamed to match any other 1-regular graph, as there will always be $\frac{n}{2}$ disconnected components of 2 adjacent vertices. Thus there is $\boxed{1}$ distinct (up to isomorphism) 1-regular graph for all even n.

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4.4 Part 4 **4 / 4**

- ✓ **+ 4 pts** Correct
- **+ 1 pts** not answering this question
- **+ 2 pts** uses stuff we didn't talk about
- **+ 3 pts** missing detail
- **+ 0 pts** blank/ no progress towards solution.
- **+ 3 pts** some mistake in understanding what "isomorphic means"
- **+ 0 pts** Click here to replace this description.

(3) If a graph G is 1-regular with n vertices, since $\sum \deg(v) = 2E$, there are $\frac{n}{2}$ edges. However, there must be an integer amount of edges, and thus n must be even.

(4) All 1-regular graphs with n-vertices will have $\frac{n}{2}$ edges each connected to two vertices. Thus any 1-regular graph can be renamed to match any other 1-regular graph, as there will always be $\frac{n}{2}$ disconnected components of 2 adjacent vertices. Thus there is $\boxed{1}$ distinct (up to isomorphism) 1-regular graph for all even n.

(5) Using the results from parts (1) and (2), since G_1 and G_2 are isomorphic if and only if $\bar{G}_1$ and $\bar{G}_2$ are isomorphic, and G is r-regular if and only if $\bar{G}$ is $n-r-1$ regular, r-regular graphs G_1 and G_2 are only isomorphic if and only if $n-r-1$ regular graphs $\bar{G}_1$ and $\bar{G}_2$ are isomorphic. Thus we can see that the amount of distinct (up to isomorphism) r-regular graphs is equal to the amount of distinct (up to isomorphism) $n-r-1$ regular graphs. Thus, the amount of $n-2$ regular graphs with n vertices is equal to the amount of $n-n+2-1$ regular (or 1-regular) graphs. Using the result of part (4), there is $\boxed{1}$ distinct (up to isomorphism) $n-2$ regular graphs with n vertices, when n is even.

4.5 Part 5 **2 / 2**

- ✓ **+ 2 pts** **Correct/ correct based on answer to previous question**
- **+ 0 pts** blank/ no progress towards the solution
- **+ 1 pts** some misunderstanding about isomorphisms
- **+ 1 pts** missing justification/ problems with justification

5) (1) Since the people who are infected start infecting 2 new people 2 days after they catch the disease, the amount of infected people at day n is equal to the previous days amount added to twice the amount of people who have had it for at least 2 days and can infect new people.

Amount of people infected during the previous day = S_{n-1}
Amount of people getting infected by people having it ≥2 days = $2S_{n-2}$

Thus $\boxed{S_n = S_{n-1} + 2S_{n-2},\ S_0 = 0,\ S_1 = 1}$

(2) Polynomial form → $t^2 = t + 2$

$t^2 - t - 2 = 0$

$(t-2)(t+1) = 0$

$t = 2, -1$

General Form → $S_n = b(2)^n + d(-1)^n$

$S_0 = 0 = b + d$, $S_1 = 1 = 2b - d$

Thus $1 = 3b$, $b = 1/3$, $d = -1/3$

Thus, $\boxed{S_n = \frac{1}{3}(2)^n - \frac{1}{3}(-1)^n}$

(3) Since people recover after their 10^{th} day of the infection, we can subtract the amount of people who've had the disease >10 days from the initial recurrence relation.

Amount of people who've had the disease >10 days = S_{n-10}

Thus the new recurrence is

$\boxed{S_n = S_{n-1} + 2S_{n-2} - S_{n-10},\ S_{<0} = 0,\ S_1 = 1}$

5.1 Part 1 5 / 5

- ✓ + 3 pts Having idea of cases to figure out people sick on nth day
- ✓ + 1 pts Fully correct relation
- ✓ + 1 pts Initial conditions

5) (1) Since the people who are infected start infecting 2 new people 2 days after they catch the disease, the amount of infected people at day n is equal to the previous days amount added to twice the amount of people who have had it for at least 2 days and can infect new people.

Amount of people infected during the previous day = S_{n-1}
Amount of people getting infected by people having it ≥ 2 days = $2S_{n-2}$

Thus $\boxed{S_n = S_{n-1} + 2S_{n-2},\ S_0 = 0,\ S_1 = 1}$

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Amount of people who've had the disease >10 days = S_{n-10}

Thus the new recurrence is

$$\boxed{S_n = S_{n-1} + 2S_{n-2} - S_{n-10},\ S_{<0} = 0,\ S_1 = 1}$$

5.2 Part 2 **2 / 2**

- ✓ **+ 2 pts** **Correct, based on 1**
- **+ 1 pts** Small mistake, based on 1
- **+ 0 pts** No points

5) (1) Since the people who are infected start infecting 2 new people 2 days after they catch the disease, the amount of infected people at day n is equal to the previous days amount added to twice the amount of people who have had it for at least 2 days and can infect new people.

Amount of people infected during the previous day $= S_{n-1}$
Amount of people getting infected by people having it ≥ 2 days $= 2S_{n-2}$

Thus $\boxed{S_n = S_{n-1} + 2S_{n-2}, \; S_0 = 0, S_1 = 1}$

(2) Polynomial form $\rightarrow t^2 = t + 2$

$$t^2 - t - 2 = 0$$
$$(t-2)(t+1) = 0$$
$$t = 2, -1$$

General Form $\rightarrow S_n = b(2)^n + d(-1)^n$

$S_0 = 0 = b + d$, $S_1 = 1 = 2b - d$

Thus $1 = 3b$, $b = 1/3$, $d = -1/3$

Thus, $\boxed{S_n = \frac{1}{3}(2)^n - \frac{1}{3}(-1)^n}$

(3) Since people recover after their 10^{th} day of the infection, we can subtract the amount of people who have had the disease >10 days from the initial recurrence relation.

Amount of people who've had the disease >10 days $= S_{n-10}$

Thus the new recurrence is

$$\boxed{S_n = S_{n-1} + 2S_{n-2} - S_{n-10}, \; S_{<0} = 0, S_1 = 1}$$

5.3 Part 3 **1 / 3**

- ✓ **+ 0.5 pts** **Idea of subtracting sick people from 10 days ago**
- ✓ **+ 0.5 pts** **Correct indexing of n - 10 (or n - 12 if correct)**
- **+ 0.5 pts** Knew that - s_{n - 10} was not correct, tried to subtract only people who got sick on the 10th previous day
- **+ 0.5 pts** Fully correct
- **+ 1 pts** Give enough initial conditions
- **+ 0 pts** No points

(4) Conceptually, we can see that the original recourrence relation almost doubles the amount of infected people per day. This indicates that S_{n-10} is large magnitudes smaller than S_{n-1} and S_{n-2}. Thus the addition of S_{n-1} and S_{n-2} will always outweigh the subtraction of S_{n-10}, and so S_n will always increase.

5.4 Part 4 **0 / 5**

- ✓ **+ 0 pts** **No points**
- **+ 3 pts** Idea to use (strong) induction
- **+ 1 pts** Handled base cases correctly, if using induction
- **+ 1 pts** Inductive step correct
- **+ 2 pts** Daily analysis of people recovering always infecting 2 more people
- **+ 3 pts** If doing a daily analysis, then noting that (besides base cases) 10 days ago there were people infected OR doing a correct dichotomy

6) Suppose, by contradiction that a planar embedding of a connected planar graph has an odd number of faces that are bounded by cycles of odd lengths. We know that the euler characteristic holds for connected planar graphs. We can transform the connected graph G into the simple connected graph G' by deleting any parallel edges (leaving one of the edges to maintain adjacency). This removes only faces that are bound by even cycles since parallel edges form a face bordered by the 2 parallel edge. This transformation also preserves all other cycles in the graph that bound a face. Now, we can remove all even cycles from G', as removing the edges that form a cycle that borders a face only removes that face and cannot create a new face. Now we are left with only the faces bounded by odd cycles, of which we have an odd amount by the initial assumption. Thus, we can remove 2 odd cycles per iteration (which preserves the assumption of having an odd number of faces bounded by odd cycles) until we are left with only 1 odd cycle. However, this 1 odd cycle must be the 2-regular graph with n vertices (as there is only one cycle in the simply connected graph) where n is odd. This graph is known to have 2 faces, the inside and outside face, which contradicts the initial assumption. Thus the planar embedding of a connected planar graph has an even number of faces bounded by odd cycles.

6 Question 6 **2 / 5**

+ **2.5 pts** Every edge is on the boundary of 2 faces or 0 faces
+ **2.5 pts** Therefore sum of face boundary lengths must be even

✓ + **2 pts** **(partial) Argument has some correct statements but contains a few errors**

+ **1 pts** (partial) Argument has some correct statements but contains errors.
+ **0 pts** Blank or incorrect

2 This process can affect the faces with odd boundary length too. What if you have an edge that is a part of 2 cycles, one of odd length and one of even length?

7) (1) Let the 6 coins be numbered 1,2,3,4,5,6

Step 1: Weigh coins {1,2} against {3,4}

Case 1: {1,2} is heavier

Step 2: Weigh {1,2} against {5,6}

Case 1.1: {1,2} is heavier

Step 3: Weigh 1 against 5

Case 1.1.1: 1 is heavier

Result: 1 is heavier than the others

Case 1.1.2 They are balanced.

Result: 2 is heavier than the others.

Case 1.2: They are balanced

Step 3: Weigh 1 against 3

Case 1.2.1: 3 is lighter

Result: 3 is lighter than the others

Case 1.2.2: They are balanced

Result: 4 is lighter than the others

Case 2: {3,4} is heavier

Step 2: Weigh {3,4} against {5,6}

Case 2.1: {3,4} is heavier

Step 3: Weigh 3 against 5

Case 2.1.1: 3 is heavier

Result: 3 is heaver than the others

Case 2.1.2: They are balanced

Result: 4 is heavier than the others.

Case 2.2: They are balanced

Step 3: Weigh 3 against 1

Case 2.2.1: 1 is lighter

Result: 1 is lighter than the others

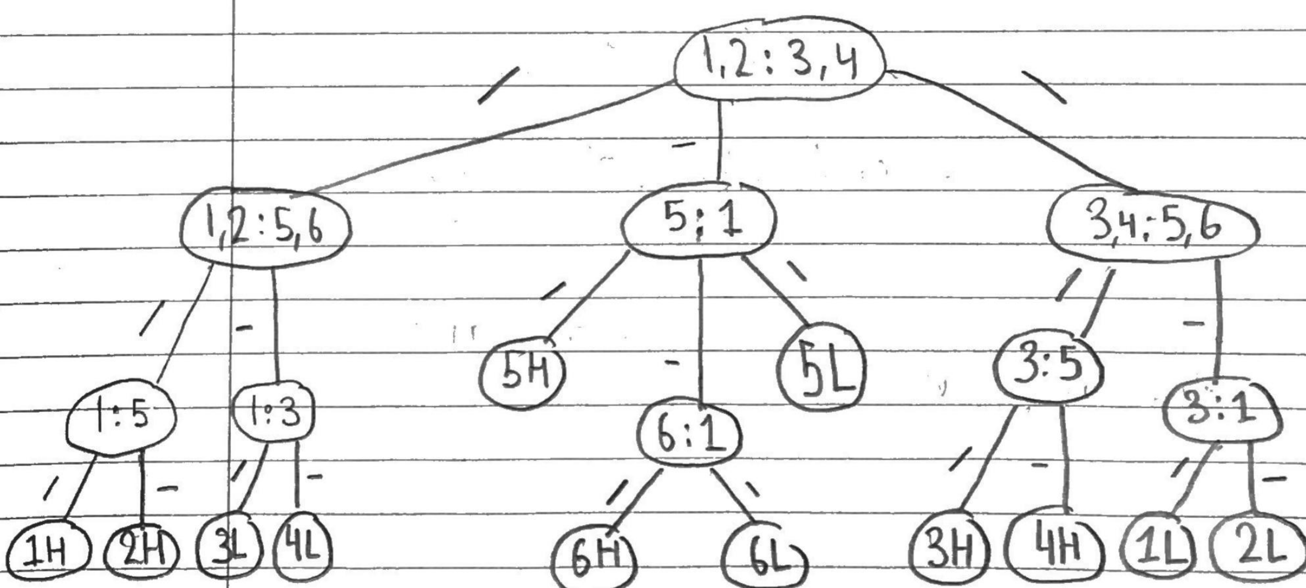
Case 2.2.2: They are balanced

Result: 2 is lighter than the others

Algorithm Continued on next page

Case 3: They are balanced
Step 2: Weigh 5 against 1
Case 3.1: 5 is heavier
Result: 5 is heavier than the others
Case 3.2: 5 is lighter
Result: 5 is lighter than the others
Case 3.3: They are balanced
Step 3: Weigh 1 against 6
Case 3.3.1: 6 is heavier
Result: 6 is heavier than the others
Case 3.3.2: 6 is lighter
Result: 6 is lighter than the others

This algorithm can be represented as a ternary tree:



7.1 Part 1 **3.5 / 5**

- ✓ **+ 3.5 pts** **Algorithm correctly finds the odd coin and its weight**
- **+ 1.5 pts** Correct explanation why algorithm works
- **+ 0 pts** Blank or incorrect
- **+ 2 pts** (partial) Algorithm finds the odd coin but not whether it is lighter or heavier
- **+ 1 pts** (partial) Algorithm distinguishes between most cases

(2) The fewest amount of weighing that guarantees a solution is the height of the decision tree that represents the algorithm, which in my case is 3. The decision tree that models the algorithm is a ternary tree.

We know from the release sample final that in a ternary tree of height h and terminal vertices k, $k \leq 3^h$.

Pf) Using induction on h:

Base case: $h=0 \to 1 \leq 3^0$ ✓

Suppose that $k \leq 3^h$ for all $0 \leq h_i < h$

Thus the number of k for a ternary tree of height h is the children of the three subtrees of the tree. All three subtrees have $h_1, h_2, h_3 \leq h-1$. By our inductive hypothesis we see that they have terminal vertices $\leq 3^{h_1}, 3^{h_2}, 3^{h_3}$

Thus k for height $h \leq 3^{h_1} + 3^{h_2} + 3^{h_3} \leq 3(3^{h-1}) \leq 3^h$ □

Thus, since there are 12 possible outcomes of the algorith, the number of weighs required must be greater than $\log_3 12$. The smallest integer than satisfies that is 3. Thus the minimum amount of weighs to solve this problem is 3, so my algorithm is the best possible solution.

7.2 Part 2 **5 / 5**

- ✓ **+ 2 pts There are 12 possible cases**
- ✓ **+ 1 pts Algorithm is modeled by 3-ary decision tree**
- ✓ **+ 2 pts Tree of height 2 can only have 9 leaves**
- **- 2 pts** Algorithm from (a) takes more than 3 weighings
- **+ 0 pts** Blank or incorrect

8) Suppose we are constructing a MST using Prim's algorithm. Let edge e' be the edge between vertices v and w. Edge e cannot also be between v and w as the graph is simple.
When Prim's algorithm reaches v (or w), either edge e is already in the tree, or the vertices to e is incident to have not been reached yet. In both cases, e' is the edge with the lowest weight not yet in the tree, which means that Prim's algorith will add it next to the MST. Thus, every MST will contain e'.

8 Question 8 **2.5 / 5**

+ **1 pts** Consider an MST that doesn't contain $$e'$$
+ **2 pts** Adding $$e'$$ to this MST must create a cycle
+ **1 pts** This new cycle has length at least 3
+ **1 pts** So there is an edge in this cycle of weight more than that of $$e'$$, a contradiction

✓ + **2.5 pts** **(partial) Different argument that makes some progress (might include some of the items from above, but fewer then 2.5 points worth)**

+ **1 pts** (partial) Different argument makes a little progress
+ **0 pts** Blank or incorrect

3 This is a good argument that any MST made through Prim's algorithm will contain e'. But how do we know that every MST can be constructed in this way?

9) Let us consider the adjacency matrix of the subgraphs of K_n. This matrix is of size $n \times n$, constructed from only 1s and 0s. This gives us n^2 elements in the matrix. However, the major diagonal of all adjacency matrix have to be all 0s, as a vertex cannot be adjacent to itself. Thus, there are n^2-n changeable elements of the matrix. However, changing a value at the ij^{th} position also changes the value at the ji^{th} position, as the edge connects both vertices to each other. Thus there are $\frac{n^2-n}{2}$ changeable options (which can be either 0 or 1), so the total number of subgraphs of K_n (up to isomorphism) is given by $2^{\frac{n^2-n}{2}}$, by the multiplication principle. We know, from class discussion, that the amount of symetric and reflexive relations on a set with n elements is $2^{\frac{n^2-n}{2}}$ (we used this fact to find the amount of symmetric relations ∪ reflexive relations on a set. Since the amount of symmetric relations is equal to the amount of subgraphs of K_n, there must exist a bijection between them.

A possible bijection: Let R be the set of symetric and reflexive relations on a set with n elements where each element of the set is the label of a vertex in K_n. Let A_{ij} represent the value of the element in the adjacency matrix of a subgraph of K_n.

$$f(R) = \begin{cases} A_{vv} = 0 \\ A_{vw} = 0 \text{ if } (v,w) \notin R \\ A_{vw} = 1 \text{ if } (v,w) \in R \end{cases}$$

Every possible adjacency matrix has a unique relation mapping to it, so the relation f is a bijection.

9 Question 9 **5 / 5**

- ✓ **+ 1 pts** **Function has correct domain and codomain.**
- ✓ **+ 2 pts** **Correct idea for the function**
- ✓ **+ 2 pts** **Argued that function is invertible, or that function is injective and surjective**
- **+ 1.5 pts** (partial) Said size of set was $$2^{\binom{n}{2}} = 2^{\frac{n(n-1)}{2}}$$
- **+ 0 pts** Blank or incorrect

10) (i) Let f be the relation mapping non full binary trees T to X_1. Since a full binary with n terminal vertices must have $n-1$ internal vertices (as there are $2n-1$ total vertices), T must have $n-1$ vertices as only terminal vertices are added in f.

If $f(T_2) = f(T_1)$, then removing all the terminal vertices from $f(T_1)$ and $f(T_2)$ results in the same tree, which implies $T_1 = T_2$. Thus, f is a one-to-one function.

Additionally, let U be a full binary tree with n terminal vertices, which implies $U \in X_1$. For each $U \in X_1$, removing those n terminal vertices produces a binary tree t with $n-1$ vertices, which implies $t \in T$. This means that every $U = f(t)$, and the function is therefore onto.

Since the function is one-to-one and onto, it is a bijection and thus $|X_1| = |T|$. This means that $|X_1|$ is the amount of nonisomorphic $n-1$ vertex binary trees, which is the $(n-1)^{th}$ Catalan number. Thus, $|X_1| = C_{n-1}$.

We can then see from figures 1-3 that X_2 consists of the trees in X_1 with 1 terminal vertex added, and one terminal vertex that is marked. Thus we know that X_2 is the set of full binary trees with $n+1$ terminal vertices, which is $X_1|_{n+1} = C_n$. There are $n+1$ terminal vertices that can be marked, so we can conclude that $|X_2| = (n+1)C_n$

10.1 Part 1 **5 / 5**

✓ **- 0 pts** correct

(2) We have shown in part (1) that the full binary tree with n terminal vertices has $n-1$ internal vertices, and so $2n-1$ total vertices. In converting this tree from X_1 to X_T, we choose one of those vertices and add either a left marked child or a right marked child. Since there are 2 possibilitys for $2n-1$ vertices, the total amount of trees that can be formed in X_T is $2(2n-1)$.

Thus, $|X_T| = 2(2n-1)$.

(3) The definition of a partition is that no item can be in more than one set of the partition, and that the union of the set in the partition is the whole partition.

Suppose that $T \in X_{T_1}$ and $T \in X_{T_2}$. This implies that the marked vertex m in T was added in 2 different ways to 2 previous graphs. However, there is only one way to add any m to a tree. Thus this implies that $X_{T_1} = X_{T_2}$. This indicates that it is not possible for an element to be in more than one set in $\{X_T : T \in X_1\}$.

We know that X_T is the set of all $n+1$ terminal vertices tree constructed from $T \in X_1$ which is a full binary tree with n terminal vertices. This indicates that all members of all X_T are in X_2. Thus the union of $X_T : T \in X_1 \subseteq X_2$. However, we also know that when we remove a marked vertex from $x \in X_2$, it is a full binary tree with n terminal vertices, which means it is a member of some X_T. Thus $X_2 \subseteq$ the union of $X_T : T \in X_1$. This indicates that the union of $X_T : T \in X_1 = X_2$.

Since the union of the set is equal to X_2, and no element can be in more than one set, $\{X_T : T \in X_1\}$ is a partition of X_2.

10.2 Part 2 **3 / 5**

✓ **- 2 pts** **Big mistake or missing that these different graphs are all unique.**

(2) We have shown in part (1) that the full binary tree with n terminal vertices has $n-1$ internal vertices, and so $2n-1$ total vertices. In converting this tree from X_1 to X_T, we choose one of those vertices and add either a left marked child or a right marked child. Since there are 2 possibilitys for $2n-1$ vertices, the total amount of trees that can be formed in X_T is $2(2n-1)$.

Thus, $|X_T| = 2(2n-1)$.

(3) The definition of a partition is that no item can be in more than one set of the partition, and that the union of the set in the partition is the whole partition.

Suppose that $T \in X_{T_1}$ and $T \in X_{T_2}$. This implies that the marked vertex m in T was added in 2 different ways to 2 previous graphs. However, there is only one way to add any m to a tree. Thus this implies that $X_{T_1} = X_{T_2}$. This indicates that it is not possible for an element to be in more than one set in $\{X_T : T \in X_1\}$.

We know that X_T is the set of all $n+1$ terminal vertices tree constructed from $T \in X_1$ which is a full binary tree with n terminal vertices. This indicates that all members of all X_T are in X_2. Thus the union of $X_T : T \in X_1 \subseteq X_2$. However, we also know that when we remove a marked vertex from $x \in X_2$, it is a full binary tree with n terminal vertices, which means it is a member of some X_T. Thus $X_2 \subseteq$ the union of $X_T : T \in X_1$. This indicates that the union of $X_T : T \in X_1 = X_2$.

Since the union of the set is equal to X_2, and no element can be in more than one set, $\{X_T : T \in X_1\}$ is a partition of X_2.

10.3 Part 3 **6 / 8**

✓ **- 2 pts** smaller mistake showing that they are disjoint

(4) Since $\{X_T ; T \in X_1\}$ is a partition of X_2, all $X_{T_1} \cap X_{T_2} = \emptyset$.

Thus $|X_2| = \sum_{T \in X_1} |X_T|$

From part (2), we get $|X_T| = 2(2n-1)$

Thus $|X_2| = 2(2n-1) \sum_{T \in X_1}$

$$\boxed{|X_2| = 2(2n-1)|X_1|}$$

Continuing with the conclusions from part (1)

$|X_2| = (n+1)C_n$, $|X_1| = C_{n-1}$

Thus $(n+1)C_n = 2(2n-1)C_{n-1}$

$$\boxed{C_n = \frac{2(2n-1)}{(n+1)} C_{n-1}}$$

10.4 Part 4 2 / 2

✓ + **2 pts** Correct

+ **1 pts** some problem with using the partition

+ **0 pts** blank