Final

Name: Hyounjun Chang

Student ID: 105-132-882

Section:

	Tuesday:	Thursday:	
	1A	1B	TA: Albert Zheng
	1C	1D	TA: Benjamin Spitz
	1E	1F	TA: Eilon Reisin-Tzur

Instructions: Do not open this exam until instructed to do so. Please print your name and student ID number above, and circle the number of your discussion section. **You may not use calculators,** books, notes, or any other material to help you. Please make sure **your phone is silenced** and stowed where you cannot see it. Remember that you are bound by a conduct code.

Please get out your id and be ready to show it during the exam.

Please do not write below this line.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
Total:	90	

1. (10 points) Circle the correct answer (only one answer is correct for each question)

   1. $\frac{n!}{k!(n-k)!} + \frac{n!}{(k+1)!(n-k-1)!} =$ $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$

      (a) $\frac{(n+k)!}{k!n!}$ ✗

      (b) $\frac{(n+1)!}{k!(n+1-k)!}$

      (c) $\frac{(n+1)!}{(k+1)!(n-k)!}$ ✗

      (d) none of the above

   2. The decision tree of a sorting algorithm for sorting n items (where at each step we can only decide whether or not one item is less than other) necessarily has:

      (a) a height of $\geq \lg(n!)$

      (b) a height of $\Omega \lg(n!)$ (but not necessarily a height of $\geq \lg(n!)$)

      (c) a height of $O(\lg(n!))$ ✗

      (d) a height of $O(n \lg n)$ ✗

   3. If G is a graph with n vertices and $n-2$ edges, then:

      (a) G is a tree

      (b) G is connected

      (c) G is disconnected

      (d) G is simple

Question 1 continues on the next page...

Question 1 continued...

4. Which of these graphs has an Euler cycle?

   (a) K_4

   (b) K_5 → degree 4 each

   (c) $K_{3,3}$ ×

   (d) $K_{2,3}$ ×

5. What is the *fewest* number of edges (i.e. in the best case) that could be examined by Dijkstra's algorithm on a graph with n vertices? (We examine edges in the part of the algorithm where we update labels.) You answer should be true for all n.

   (a) Less than or equal to n

   (b) More than n but less than or equal to $n^2/2$

   (c) More than $n^2/2$ but less than or equal to n^2

   (d) More than n^2

2. In this question write down your answer, no need for any justification. Leave your answers in a form involving factorials, $P(n,m)$, $\binom{n}{m}$, exponents, etc. $x^2 = x + 6$ $x^2 - x - 6 = 0$ $(x-3)(x+2)$

(a) (2 points) If $s_n = s_{n-1} + 6s_{n-2}$ and $s_0 = 2$, $s_1 = 1$, what is s_{100}?

$a3^n + b(-2)^n$

$a + b = 2$ $a=1$ $b=1$

$3a - 2b = 1$

$3^{100} + (-2)^{100}$

(b) (2 points) How many ways can 7 distinct math majors and 4 distinct CS majors sit in a circle, if the CS majors won't sit by each other and we say that two seatings are the same if they are related by a rotation?

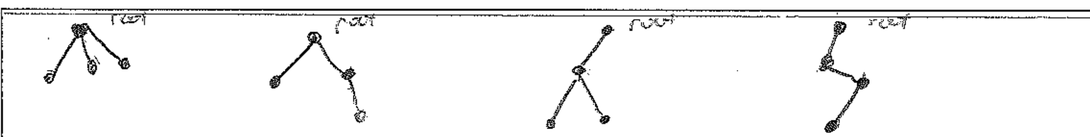
_ 1 _ 1 _ 1

$\frac{7!\,P(7,4)}{11}$

(c) (2 points) A squirrel has 20 identical acorns that she is going to hide among 5 distinct holes. In how many ways can the squirrel hide the acorns? $x_1 + x_2 + x_3 + x_4 + x_5 = 20$ $x \geq 0$

$\binom{24}{4}$

(d) (2 points) Draw all the distinct (up to isomorphism) rooted trees with 4 vertices. Please put the root at the top.



(e) (2 points) What is the number of relations that are symmetric or reflexive on a set with n-elements?

4^n

$2^{\frac{n(n+1)}{2}}$

$1 +$

$2^n + 2^{\frac{n(n+1)}{2}}$

3. Consider the relation on the real numbers defined by $C = \{(x, y) \in \mathbb{R}\times\mathbb{R} : x - y \in \mathbb{Z}\}$.

(a) (4 points) Show that C is an equivalence relation.

Reflexitivity:

$x - x = 0$ and $0 \in \mathbb{Z}$ so xRx

Symmetry

If xRy, then $x - y = z$, $z \in \mathbb{Z}$

then $y - x = -(x-y) = -z$ and $-z \in \mathbb{Z}$

so if xRy, then yRx

transitive:

If xRy and yRz, $x - y = t_1$ $y - z = t_2$

$t_1, t_2 \in \mathbb{Z}$, then $z = y - t_2$

$x - z = x - (y - t_2) = x - y + t_2$

$= t_1 + t_2$ and $t_1 + t_2 \in \mathbb{Z}$ since $t_1, t_2 \in \mathbb{Z}$

so If xRy and yRz, xRz

$\square$

$y-z$
$y = t_2 + z$

Question 3 continues on the next page...

(b) (4 points) Let $\tilde{\mathbb{R}}$ denote the set of equivalence classes of C, i.e. $\tilde{\mathbb{R}} = \{[x] : x \in \mathbb{R}\}$. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x + 1/2$.

Show that the relatation $\tilde{f}$ from $\tilde{\mathbb{R}}$ to $\tilde{\mathbb{R}}$ defined by $\tilde{f} = \{([a],[b]) \in \tilde{\mathbb{R}} \times \tilde{\mathbb{R}} : f(a) = b\}$ is a function.

$f(x) = x + 1/2$ is a function.

equivalent class $C \Rightarrow$ 2 numbers have same decimals.

Let $\tilde{f}([a]) = [b]$ and $\tilde{f}([a]) = [c]$

then decimal place of $a + 1/2$ = decimal place of b

decimal place of $a + 1/2$ = decimal place of c

decimal place of b = decimal place of C.

And since we defined equivalent class as same decimal places,

$[b] = [c]$. so $\tilde{f}$ is a function

$\square$

(c) (2 points) Give an example of a function $g : \mathbb{R} \to \mathbb{R}$ so that the relation $\tilde{g}$ from $\tilde{\mathbb{R}}$ to $\tilde{\mathbb{R}}$ defined by $\tilde{g} = \{([a],[b]) \in \tilde{\mathbb{R}} \times \tilde{\mathbb{R}} : g(a) = b\}$ is **not** a function. (Be sure to justify your answer.)

$g(x) = \frac{1}{2}x$

ex) 1 and 2 are in same equivalence class

$g(1) = 1/2$ $g(2) = 1$

$1 - 1/2 = 1/2$, 1 and $1/2$ are not in same equivalence class!!. (this doesn't lead to same decimals for all eq. classes!)

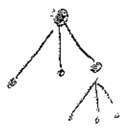
internal external

+1 -1

+3 2

4. For m a positive integer, a *full m-ary tree* is a rooted tree where every parent has exactly m children.

3



(a) (5 points) If T is a full m-ary tree with i internal vertices, how many terminal vertices does T have?

$(m-1)i+1$ terminal vertices:

every time we make terminall vertice to internal vertice we add $(m-1)$ terminal verticies and 1 internal vertice.

If $i=0$, there is 1 terminal vertice. (just root)

(b) (5 points) Show that if T is a full m-ary tree of height h with t terminal vertices, then $t \leq m^h$.

To get the minimum height, every single vertice that is not on the Lowest level has m children.

then, at the lowest Level h, number of vertices will be m^h. (m^1 in 1st level, m^2 in 2nd level, etc.)

Suppose that one of vertices NOT on the last level is a terminal vertice. Then, this has $m^h - m^{(h-l)}$ where l is level of vertice, and this is less then m^h.

And by induction, taking more vertices from bottom layer reduces terminal vertices.

Suppose $t = m^h$ (worst case scenario)

$m^h = m^h$ if only last level has terminal vertices

If there is terminal vertice above "level h", and we want to keep it that way, to add $m^{(h-1)}$ vertices we have to add vertices at $(h+k)$ level, so, height of tree is increased to $h_f > h_{orig}$.

$h_{orig} = h$

(other cases) so and $t \leq m^{h_f}$ since $h_f > h_{orig}$. If t is between $m^{h-1} < t \leq m^h$, then the height must be at least h, so $t \leq m^h$. (can't fit t vertices into m^{h-1})

5. (a) (6 points) Show that if G is a connected weighted graph where all the edges of G have distinct weights then G has a unique minimal spanning tree.

Proof with prim's algorithm.

- Prim's algorithm ALWAYS gives weight of minimum spanning trees.

Suppose there are t edges, and n vertices we choose $(n-1)$ edges from t.

- all edges have different weights we can add.

$E_1 < E_2 < E_3 \cdots \cdots < E_t$

E_k = kth smallest edge.

Suppose E_k was largest edge that was chosen By Prim's algorithm. (there maybe an edge $E_1 < E_{part} < E_k$ which may not have been chosen).

Base case ⇒ Suppose $k = n-1$. Then it has the smallest sum possible, so it's unique.

Induct on number k. (we leave one number out and add $k+1$)

Suppose on $k+1$.

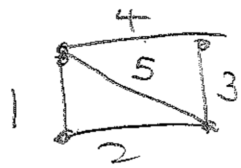
- For each vertice not chosen, there is an edge with less weight that still makes a spanning tree with less weight.
- Suppose you can replace the not chosen edge $E_1 < E_{part} < E_k$ which keeps it connected. (with any edge chosen).
- We can't replace the edge with any higher weights in the binary tree since it would have even less weight (or we get a new weight of min spanning tree)
- We can't replace the unchosen edge with anything with less weight since the weight will be heavier.

Question 5 continues on the next page...

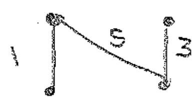
(b) (4 points) Give an example of a connected weighted graph G so that all the edges of G have distinct weights and G has at least two distinct spanning trees that have the same total weight, i.e. the sums of the weights of the edges in these two distinct trees agree, *or* prove that no such weighted graph exists.

Two Spanning trees can have same weight

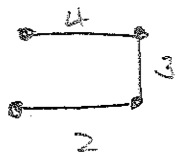
ex)



ex.



$1+5+3=9$



$4+3+2=9$.

6. (a) (3 points) Show that for G a connected simple planar graph containing a cycle if G has E edges and F faces, then $2E \geq 3F$.

- Simple graph, so it has no repeating edges between 2 vertices, and no loop.
- Shortest cycle is length 3.
- In planar graph, every edge is bondary to at most 2 faces.
- We have a cycle, so $F \geq 2$, and a cycle is required for additional face from first one. (this doesn't work all time with $F=1$)

So $2E \geq 3F$, since each face requires 3 edges, and each edge can be bondary for at most 2 faces. (Not all edges may be a bondary)

(b) (3 points) Show that for G a connected simple planar graph containing a cycle if G has E edges and V vertices, then $E \leq 3V - 6$.

Euler's formula: $V - E + F = 2$ for planar graphs

$V = E - F + 2 \Rightarrow E \leq 3(E - F + 2) - 6$ ← equivalent form

$E \leq 3E - 3F + 6 - 6$

$\cancel{E \leq 3E - 3F}$

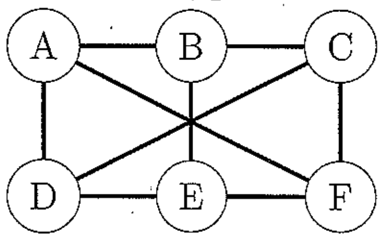
$3F + E \leq 3E$

$3F \leq 2E$ ← this was proved in part (a), and is equivalent expression

So $E \leq 3V - 6$

Question 6 continues on the next page...

(c) (4 points) Is the following graph planar? If it is give a planar drawing of it. If not, prove that it is not planar.



No, this is a bitartite graph $K_{3,3}$.

1st partition: A C E

2nd partition: B D F.

$K_{3,3}$ is not planar →

Cycle: → at least length 3.

If planar, then $V - E + F = 2$, AND

$2E \geq 3F$. V =

If planar, $6 - 9 + F = 2$ so $F = 5$

$2(6) \geq 3(5)$. $12 \geq 15$

↑ this is not true

$K_{3,3}$ is not planar

So this graph is NOT planar.

□

7. (a) (5 points) Show that for all $n \geq 1$, $7^n - 1$ is divisible by 6.

Base case: $n=1$ $7-1=6$; $6 = 1 \cdot 6$ is divisible by 6.

Inductive Step: suppose $7^n - 1$ is divisible by 6.

then.. $7^{(n+1)} - 1 = 7(7^n) - 1 =$

$7(7^n - 1) + 7 - 1 = 7(7^n - 1) + 6$

$7(7^n - 1)$ is divisible by 6 since $7^n - 1$ is divisible by 6.

6 is divisible by 6 $6 = 1 \cdot 6$.

So.. $7(7^n - 1) + 6$ is divisible by 6, .. so

$7^{n+1} - 1$ is divisible by 6.

5

(b) (5 points) Show that there is a number of the form $\sum_{i=0}^{n} 10^i$ (i.e. a number consisting only of 1s) that is divisible by 7.

NOT DONE

Divisible by 7 means $X \% 7 = 0$.

Since 10 doesn't have 7 in prime factorization,

$10^i \% 7 \neq 0$. We are adding the modulus of 10^i,

and if the sum of modulus is divisible by 7, then the number is divisible by 7.

13 Let $10^n \% 7 = k$. $10^{n+1} \% 7 = (10^n \% 7 \times 10) \% 7$

$= (10k) \% 7$ There is a pattern in the remainders.

30/7= $1 \Rightarrow 3 \Rightarrow 2 \Rightarrow 6 \Rightarrow 4 \Rightarrow 5 \Rightarrow$ back to 1, and repeat

$1+3+2+6+4+5 = 7(3) = 21$, which is divisible

60 by 7. (sum of modulus is divisible by 7)

So.. There is a number $\sum_{i=0}^{n} 10^i$ where sum of remainders of 10^i

are divisible by 7, so There is a number $\sum_{i=0}^{n} 10^i$ divisible by 7.

8. A *balanced binary tree* is a binary tree where for each vertex the heights of the left and right subtrees of that vertex differ by at most one. Let v_n denote the minimum number of vertices in a balanced binary tree of height n.

(a) (4 points) Show that v_n satisfies for $n \geq 2$ the recurrence $v_n = v_{n-1} + v_{n-2}$. $+1$

Height can differ by one: Start with root. Suppose
left subtree has height $n-1$ (from left child of root)
right subtree has height $n-2$ (from right child of root)

To get min. num of vertices, we add the minimum of subtrees + 1 (for the root).

So $\underbrace{V_n}{} = \underbrace{V_{n-1}}_{\text{"left subtree"}} + \underbrace{V_{n-2}}_{\text{"right subtree"}} + \underbrace{1}_{\text{root}}$

(b) (3 points) Show that for $n \geq 0$, $v_n = \cancel{F_{n+2}}$, where F_k is the k^{th} Fibonacci number. $F_n+3 - 1$ 0 1 1 2 3

Base case

height $n=0$ $V_0 = 1$ (root). $F_3 = 2$ $F_3 - 1 = 1$ ✓ Base case holds
$n=1$ $V_1 = 2$. $F_4 = 3$ $F_4 - 1 = 2$

1
2
4
7
12

F

Suppose $V_n = F_{n+3} - 1$ ... induction

$V_{n+1} = V_n + V_{n-1} + 1 = (F_{n+3} - 1) + (F_{n+2} - 1) + 1$

$= F_{n+3} + F_{n+2} - 1$

$V_{n+1} = F_{n+4} - 1$

$\boxtimes$

Question 8 continues on the next page...

(c) (3 points) Show that $v_n = \Theta(\phi^{n+2})$, where $\phi = \frac{1+\sqrt{5}}{2}$.

Fibonacci sequence: $a_n = a_{n-1} + a_{n-2}$

$x^2 - x - 1 = 0$

$r = \frac{1 \pm \sqrt{5}}{2}$ $\quad$ $a + b = 0$

$a\left(\frac{1+\sqrt{5}}{2}\right) + b\left(\frac{1-\sqrt{5}}{2}\right) = 1$

$a\left(\frac{1+\sqrt{5}}{2}\right) - a\left(\frac{1-\sqrt{5}}{2}\right) = 1$ $\quad$ $a = \frac{1}{\sqrt{5}}$ $\quad$ $b = -\frac{1}{\sqrt{5}}$

$\sqrt{5}\, a = 1$

Fibonacci sequence $F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$

and $\left|\frac{1-\sqrt{5}}{2}\right| < 1$, so as n becomes larger, $\left(\frac{1-\sqrt{5}}{2}\right)^n$ decreases.

$V_n = F_{n+3} - 1 > \left(\frac{1}{\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - 1$

As n gets really large, V_n approaches $\left(\frac{1}{\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)\phi^{n+2}$.

Let $C_1 = 10$, (bigger exponential base)

There exists k, where $k \geq 0$ such that $10(\phi^{n+2}) \geq V_n$ since $10 > \frac{1}{\sqrt{5}}\frac{(1+\sqrt{5})}{2}$ for $n \geq k$.

So this is $O(\phi^{n+2})$

Let $C_2 = \frac{1}{1000}$ which is $\frac{1}{1000} < \frac{1(1+\sqrt{5})}{2\sqrt{5}}$

$\frac{1}{1000}(\phi^{n+2}) \leq V_n$ for all $n \geq k$. (smaller exponential base)

So it is $\Omega(\phi^{n+2})$

Since it's $O(\phi^{n+2})$ and $\Omega(\phi^{n+2})$ this is ~~Θ~~ (ϕ^{n+2})

9. (a) (4 points) Show that $\sum_{i=0}^{n} 2^i \binom{n}{i} = 3^n$.

Binomial Theorem: $(x+y)^n = \sum_{i=0}^{n} x^i y^{n-i} \binom{n}{i}$

$(2+1)^n = \sum_{i=0}^{n} 2^i \, 1^{(n-i)} \binom{n}{i}$

$(2+1)^n = \sum_{i=0}^{n} 2^i \binom{n}{i}$ $\qquad 1^{(n-i)} = 1$

so... $3^n = \sum_{i=0}^{n} 2^i \binom{n}{i}$

$\boxed{1/4}$

(b) (6 points) Show that $\binom{n+m}{r} = \sum_{i=0}^{r} \binom{n}{i}\binom{m}{r-i}$.

Combinatoric Sense:

we are choosing r things from 2 piles with size n and m

If we choose i items from pile n, we choose $(r-i)$ items from pile m.

There are $\binom{n}{i}$ ways to choose from pile n, and $\binom{m}{r-i}$ ways to choose from pile m.

number of possible combinations for each i is

$\binom{n}{i} \times \binom{m}{r-i}$ since the two piles are "independent".

We have to add the sum of this for all possible i s.

$0 \le i \le n$ since these are all possible number of items we can choose from pile n, we add the sum from $i=0$ to $i=n$.

This page has been left intentionally blank. You may use it as scratch paper. It will not be graded unless indicated very clearly here and next to the relevant question.