Math 61-1 Final exam

EUGENE LO

TOTAL POINTS

64.5 / 90

QUESTION 1

Multiple choice 10 pts

1.1 **2 / 2**

✓ **- 0 pts** **Correct (c)**

- 2 pts Incorrect

1.2 **2 / 2**

✓ **- 0 pts** **Correct (a)**

- 2 pts Incorrect

1.3 **2 / 2**

✓ **- 0 pts** **Correct (c)**

- 2 pts Incorrect

1.4 **2 / 2**

✓ **- 0 pts** **Correct (b)**

- 2 pts incorrect

1.5 **0 / 2**

- 0 pts Correct (a)

✓ **- 2 pts** **Incorrect**

QUESTION 2

Short answer 10 pts

2.1 **2 / 2**

✓ **- 0 pts** **Correct ((-2)^100 + 3^100)**

- 1 pts Almost correct (small arithmetic error in answer)

- 2 pts Incorrect

2.2 **0 / 2**

- 0 pts Correct (C(7,4)6!4!)

- 1 pts Close

✓ **- 2 pts** **Incorrect**

2.3 **2 / 2**

✓ **- 0 pts** **Correct (24C4)**

- 1 pts Close

- 2 pts incorrect

2.4 **2 / 2**

✓ **- 0 pts** **Correct**

- 1 pts Close (Three of four)

- 2 pts Incorrect

2.5 **1 / 2**

- 0 pts Correct (2^(n^2 - n) + 2^(n^2 +n / 2) - 2^(n^2 - n / 2))

✓ **- 1 pts** **Close**

- 2 pts Incorrect

QUESTION 3

Equivalence relation 10 pts

3.1 it is an equivalence relation **4 / 4**

✓ **- 0 pts** **Correct**

- 1 pts issue in transitivity

- 3 pts misunderstanding of what relation is saying

- 4 pts blank

- 2 pts misunderstanding of symmetry

- 1 pts the decimal thing isn't exactly right, e.g. -.3 is related to .7

- 0 pts Click here to replace this description.

- 1 pts issue with symmetry

3.2 defining a function **3 / 4**

- 0 pts Correct

- 4 pts blank

- 2 pts need to prove uniqueness part of function

- 2 pts missing existence part of function

- 1 pts issue with uniqueness part of function

✓ **- 1 pts** **need to consider different elements in the same equivalence class**

- 1 pts thing with decimals isn't quite right, for example -.3 and .7 are related

- 3 pts big misunderstanding of the equivalence relation or function

3.3 a function that doesn't descend 0 / 2

- 0 pts Correct

✓ **- 2 pts** **your g is not a function**

- 1 pts issue with justification

- 1 pts your g does not work

- 2 pts blank

QUESTION 4

m-ary tree 10 pts

4.1 number of internal vertices 5 / 5

✓ **- 0 pts** **Correct**

- 1 pts No/incorrect answer

- 4 pts No/incorrect justification

- 2 pts Didn't justify number of total vertices

- 3 pts "Proof by example"

- 2 pts Assumed every terminal vertex had the same height as the tree

- 5 pts Nothing

- 1 pts Forgot to account for root

- 2 pts Didn't subtract off internal vertices

4.2 height 3 / 5

- 0 pts Correct

✓ **- 1 pts** **No base case**

✓ **- 1 pts** **Didn't set up/invoke induction**

- 1 pts Backwards inductive step (didn't show inductive construction is exhaustive)

- 2 pts Compared to complete tree without showing this case is extremal

- 3 pts Assumed tree is complete / inductive construction forms complete trees from complete trees

- 1 pts Assumed all immediate subtrees have height h-1

- 4 pts "Proof by example"

- 5 pts Nothing shown / Incorrect reasoning

Very good proof, but the induction needs to be more explicit: you didn't claim to show an inductive step yet you used an inductive hypothesis without acknowledging it. You should start a proof like this by saying you'll be proving this by induction on h, then show a base case. Label the inductive step clearly as such, and make note of where the inductive hypothesis is used.

QUESTION 5

spanning trees 10 pts

5.1 unique mst 3 / 6

- 0 pts Correct

- 3 pts Appeal to Prim's or Kruskal's Algorithm (without proving it can generate any MST)

- 6 pts No / Invalid reasoning

- 3 Point adjustment

You're trying to show T is unique!!

Should be "let us suppose" or similar

Which subgraph of T'?

(-3) Did not produce desired result in all cases

5.2 non unique spanning tree 4 / 4

✓ **- 0 pts** **Correct**

- 4 pts Not an example

- 4 pts Claimed no such graph exists

- 4 pts Nothing

QUESTION 6

planar graphs 10 pts

6.1 2e > 3f 3 / 3

✓ **+ 3 pts** **Correct**

+ 2 pts >= 3 edges for each face

+ **1 pts** >= 3 edges for each face (w/ mistake)
+ **1 pts** <=2 faces for each edge
+ **0 pts** Incorrect

6.2 e<3v-6 **3 / 3**

✓ + **3 pts** **Correct**
+ **2 pts** Euler's formula
+ **1 pts** Correct application with (a)
+ **0 pts** Incorrect

6.3 nonplanar graph **3 / 4**

+ **4 pts** Correct
✓ + **3 pts** **Isomorphic to K_3,3**
+ **2 pts** Mistaken/missing ismorphism to K_3,3
+ **1 pts** E <= 2v-4 or 2E >= 4F
+ **1 pts** Other partial credit
+ **0 pts** Incorrect

QUESTION 7

10 pts

7.1 7^n-1 divisible by 6 **5 / 5**

✓ + **5 pts** **Correct**
+ **1 pts** Base case
+ **1 pts** Inductive hypothesis
+ **2 pts** factoring out a 7 in inductive step as (6+1) or adding/substracting 7
+ **1 pts** Conclusion
+ **0 pts** Incorrect

7.2 number with only 1s divisible by 7 **1 / 5**

+ **5 pts** Correct
+ **0 pts** Click here to replace this description.
+ **1 pts** Look at 8 consecutive terms
+ **1 pts** Pigeonhole remainder
+ **1 pts** 7 divides a number of the form 111..000...
+ **2 pts** This implies that 7 divides 10^k*11...
✓ + **1 pts** **Unsuccessful attempt with substantial work**

QUESTION 8

balanced binary trees 10 pts

8.1 **4 / 4**

✓ - **0 pts** **Correct**
- **2 pts** incomplete, need to describe how a height n minimal balanced binary tree is made out of ones of smaller height
- **3 pts** can't just do examples
- **4 pts** blank
- **1 pts** how are you adding in these trees/ vertices?
- **3 pts** can't do induction without using some properties of minimal balanced binary trees
- **4 pts** incorrect numbers/ equation

8.2 relationship to fibonacci numbers **2 / 3**

✓ - **0 pts** **Correct**
- **1.5 pts** that is not the recurrence/ equation for the fibonacci numbers/ minimal balanced binary trees
- **1 pts** you are assuming the desired conclusion
- **3 pts** blank
- **1.5 pts** need to use recurrence for fiboacci numbers
- **1.5 pts** missing inductive step
✓ - **1 pts** **the two recurrences aren't exactly the same, you need to account for this difference**
- **0.5 pts** error in equations
- **1 pts** need to check initial conditions

8.3 Theta **2.5 / 3**

- **0 pts** Correct
✓ - **0.5 pts** **need to account for other term in equation for fibonacci numbers (sometimes it is contributing something positive, something something negative)**
- **2 pts** wrong formula for fibonacci numbers/ v_n
- **1 pts** issue with big O
- **1 pts** issue with omega
- **3 pts** blank/ no gradable work
- **1 pts** wrong equations/ issues with constants
- **2 pts** need to use equation for v_n/ Fibonacci numbers

QUESTION 9

binomial coefficients 10 pts

9.1 3^n 4 / 4

✓ + **4 pts** **Correct**

+ **3 pts** Minor error

+ **2 pts** Binomial theorem

+ **1 pts** Attempted induction or counting argument

+ **0 pts** Incorret

9.2 vandermonde identity 0 / 6

+ **6 pts** Correct

+ **5 pts** Minor errror

+ **3 pts** One part of counting argument or (x+y)^n+m

+ **1 pts** Attempted to use induction/binomial thrm/Pascal's identity

✓ + **0 pts** **Incorrect**

Final

Name: Eugene Lo.

Student ID: 905108982

Section:

	Tuesday:	Thursday:	
	1A	1B	TA: Albert Zheng
	1C	1D	TA: Benjamin Spitz
	1E	1F	TA: Eilon Reisin-Tzur

Instructions: Do not open this exam until instructed to do so. Please print your name and student ID number above, and circle the number of your discussion section. **You may not use calculators,** books, notes, or any other material to help you. Please make sure **your phone is silenced** and stowed where you cannot see it. Remember that you are bound by a conduct code.

Please get out your id and be ready to show it during the exam.

Please do not write below this line.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
Total:	90	

1. (10 points) Circle the correct answer (only one answer is correct for each question)

1. $\frac{n!}{k!(n-k)!} + \frac{n!}{(k+1)!(n-k-1)!} =$

   (a) $\frac{(n+k)!}{k!n!}$

   (b) $\frac{(n+1)!}{k!(n+1-k)!}$

   (c) $\frac{(n+1)!}{(k+1)!(n-k)!}$

   (d) none of the above

$\frac{n!(k+1)}{(k+1)k!(n-k)(n-k-1)!} + \frac{n!(n-k)}{(k+1)(k!)(n-k-1)!(n-k)}$

$\frac{n!(k+1) + n!(n-k)}{(k+1)!(n-k)!}$

$n!(\cancel{k}+1+n-\cancel{k})$ $\quad \frac{n!(n+1)}{(k+1)!(n-k)!}$

$\frac{n+1!}{(k+1)!(n-k)!}$

2. The decision tree of a sorting algorithm for sorting n items (where at each step we can only decide whether or not one item is less than other) necessarily has:

   (a) a height of $\geq \lg(n!)$

   (b) a height of $\Omega \lg(n!)$ (but not necessarily a height of $\geq \lg(n!)$)

   (c) a height of $O(\lg(n!))$

   (d) a height of $O(n \lg n)$

$n \leq$

$n! \leq 2^h$

$\log n! \leq h$

3. If G is a graph with n vertices and $n-2$ edges, then:

   (a) G is a tree

   (b) G is connected

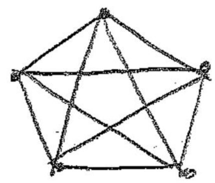
   (c) G is disconnected

   (d) G is simple

Question 1 continues on the next page...

Question 1 continued...

4. Which of these graphs has an Euler cycle?
   (a) K_4
   (b) K_5
   (c) $K_{3,3}$
   (d) $K_{2,3}$



5. What is the *fewest* number of edges (i.e. in the best case) that could be examined by Dijkstra's algorithm on a graph with n vertices? (We examine edges in the part of the algorithm where we update labels.) You answer should be true for all n.
   (a) Less than or equal to n
   (b) More than n but less than or equal to $n^2/2$
   (c) More than $n^2/2$ but less than or equal to n^2
   (d) More than n^2

$\frac{n(n-1)}{2}$

$s_n = s_{n-1} + 6s_{n-2}$

$s^2 = s + 6$ $\qquad s^2 - s - 6$ $\qquad (s-3)(s+2)$

2. In this question write down your answer, no need for any justification. Leave your answers in a form involving factorials, $P(n,m)$, $\binom{n}{m}$, exponents, etc.

$s = 3, -2$

(a) (2 points) If $s_n = s_{n-1} + 6s_{n-2}$ and $s_0 = 2$, $s_1 = 1$, what is s_{100}? $S_n = A(3)^n + B(-2)^n$

$3^{100} + (-2)^{100}$

$2 = A + B$
$1 = 3A - 2B$
$4 = 2A + 2B$
$5 = 5A$
$A = 1$
$B = 1$
$S_n = 3^n + (-2)^n$

(b) (2 points) How many ways can 7 distinct math majors and 4 distinct CS majors sit in a circle, if the CS majors won't sit by each other and we say that two seatings are the same if they are related by a rotation?

$\frac{7!}{7} = 6!$

$6! \times P(8,4)$

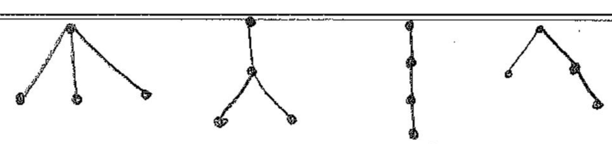
_M_M_M_M_M_M_M_

(c) (2 points) A squirrel has 20 identical acorns that she is going to hide among 5 distinct holes. In how many ways can the squirrel hide the acorns?

$x_1 + x_2 + x_3 + x_4 + x_5 = 20$

$\binom{20+5-1}{5-1} = \boxed{\binom{24}{4}}$

(d) (2 points) Draw all the distinct (up to isomorphism) rooted trees with 4 vertices. Please put the root at the top.



(e) (2 points) What is the number of relations that are symmetric or reflexive on a set with n-elements?

$2^{n^2-n} + 2^{\frac{n^2-n}{2}}$

n^2 elements in relation

2^{n^2-n}

3. Consider the relation on the real numbers defined by $C = \{(x, y) \in \mathbb{R}\times\mathbb{R} : x - y \in \mathbb{Z}\}$.

(a) (4 points) Show that C is an equivalence relation.

Reflexive: For all $(x,x) \in \mathbb{R}\times\mathbb{R}$, $(x,x) \in C$.

$x - x = 0$, and $0 \in \mathbb{Z}$, so $(x,x) \in C$. ✓

Symmetric: For $(x,y) \in \mathbb{R}\times\mathbb{R}$, if $(x,y) \in C$, then $(y,x) \in C$

$(x,y) \in C$ if $x - y \in \mathbb{Z}$

then $-(y-x) \in \mathbb{Z}$, which is just the negative of $y-x$, which is still an integer. Therefore, $(y,x) \in C$. ✓

Transitive: For $(x,y) \in \mathbb{R}\times\mathbb{R}$ and $(y,z) \in \mathbb{R},\mathbb{R}$, if $(x,y) \in C$ and $(y,z) \in C$, then $(x,z) \in C$.

If $(x,y) \in C$, then $x-y \in \mathbb{Z}$.

If $(y,z) \in C$, then $y-z \in \mathbb{Z}$

Need to show $x-z \in \mathbb{Z}$

If $x-y \in \mathbb{Z}$ and $y-z \in \mathbb{Z}$, then they should add together to form an integer (additive property of integers)

$x - \cancel{y} + \cancel{y} - z = x - z$, so

$x - z \in \mathbb{Z}$ ✓

Therefore, C is an equivalence relation.

Question 3 continues on the next page...

(b) (4 points) Let $\tilde{\mathbb{R}}$ denote the set of equivalence classes of C, i.e. $\tilde{\mathbb{R}} = \{[x] : x \in \mathbb{R}\}$. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x + 1/2$.

Show that the relatation $\tilde{f}$ from $\tilde{\mathbb{R}}$ to $\tilde{\mathbb{R}}$ defined by $\tilde{f} = \{([a], [b]) \in \tilde{\mathbb{R}} \times \tilde{\mathbb{R}} : f(a) = b\}$ is a function.

If $\tilde{f}$ is a function, then it uniquely maps every element of $[a]$ to one of $[b]$.

For $[a]\ [b], [c], [d] \in \tilde{\mathbb{R}}$

$\hat{f} = \{([a], [b]) \in \tilde{\mathbb{R}} \times \tilde{\mathbb{R}} : f(a) = b\}$ and

$\hat{f} = \{([c], [d]) \in \tilde{\mathbb{R}} \times \tilde{\mathbb{R}} : f(c) = d\}$

Consider $f(a) = a + 1/2 = b$

$f(c) = c + 1/2 = d$

If $a = c$ for $a \in [a]$ and $c \in [c]$, then

$f(c) = c + 1/2 = b$

$f(c) = c + 1/2 = d$, so $b = d$.

Therefore, $[a] \subseteq [c]$

If $c = a$ for $a \in [a]$ and $c \in [c]$, then

$f(a) = a + 1/2 = b$

$f(a) = a + 1/2 = d$, so $b = d$

Therefore, $[c] \subseteq [a]$, so $[a] = [c]$.

Therefore, the function $\tilde{f}$ defined by $\tilde{f} = \{[a], [b]\} \in \tilde{\mathbb{R}} \times \tilde{\mathbb{R}} : f(a) = b\}$ uniquely maps $[a]$ to $[b]$, and it is a function.

(c) (2 points) Give an example of a function $g : \mathbb{R} \to \mathbb{R}$ so that the relation $\tilde{g}$ from $\tilde{\mathbb{R}}$ to $\tilde{\mathbb{R}}$ defined by $\tilde{g} = \{([a], [b]) \in \tilde{\mathbb{R}} \times \tilde{\mathbb{R}} : g(a) = b\}$ is **not** a function. (Be sure to justify your answer.)

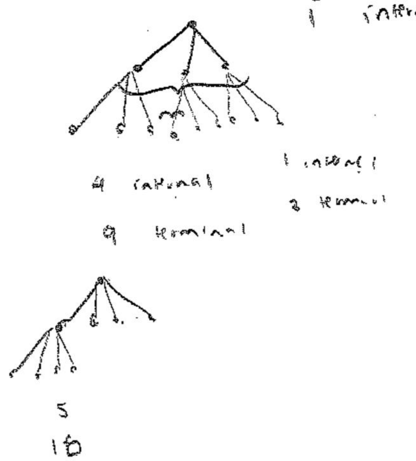
$g : \mathbb{R} \to \mathbb{R}$ $\boxed{g(x) = \sqrt{x}}$

There are some values of x, such as 5, that are not taken to a real number. $g(5) = \sqrt{5}$, and $\sqrt{5}$ is not a real number. As a result, $g : \mathbb{R} \to \mathbb{R}$ is not a function.

4. For m a positive integer, a *full m-ary tree* is a rooted tree where every parent has exactly m children.

(a) (5 points) If T is a full m-ary tree with i internal vertices, how many terminal vertices does T have?

i internal vertices



If each internal vertex has m children, then there are $im + 1$ vertices total, with 1 being the root vertex. Subtract out i internal vertices to get the # of terminal vertices of T

terminal vertices: $\boxed{im - i + 1}$

(b) (5 points) Show that if T is a full m-ary tree of height h with t terminal vertices, then $t \leq m^h$.

If an m-ary tree has a height h with t terminal vertices, then the root vertex will have m subtrees of height $h-1$, with each child of the root vertex being the root vertex of the subtree. The sum of the terminal vertices of each subtree is $\geq$ the # the terminal vertices t.

$$t \leq \underbrace{m^{h-1} + m^{h-1} + m^{h-1} + \dots + m^{h-1}}_{m \text{ terms}} = m \cdot (m^{h-1})$$
$$= m^h$$

therefore, $t \leq m^h$.

(the $\leq$ sign is for if not all branches reach the height of h; m^h is worst case).

5. (a) (6 points) Show that if G is a connected weighted graph where all the edges of G have distinct weights then G has a unique minimal spanning tree.

Let T be a unique minimal spanning tree of G. We want to prove that if G has distinct weights then G has a unique minimal spanning tree, which is T.

By contradiction, let us try to prove that T is not a unique minimal spanning tree. Say we have an edge e that is not an edge in T ($e \notin T$).

If we add e to T to form a new subgraph T', then T' must have a cycle that involves edge e. (If a tree has n vertices, it must have n-1 edges to be acyclic; if we add an edge, there will be a cycle).

The subgraph of T' contains an edge e', where $e' \in T$ and it has vertex incident on it that is also incident on e.

If we remove e', we will make a new tree T''.

Because G has distinct weights, weight(e) $\neq$ weight(e'). If weight(e) < weight(e'), then we get that weight(T'') = weight(T) + weight(e) − weight(e') < weight(T).

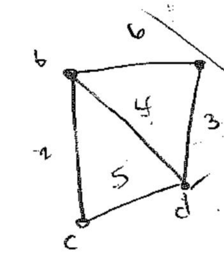
However, it is given that T is the minimum spanning tree, so weight(T) < weight(T''), so we arrive at a contradiction.

Therefore, weight(e) > weight(e') and weight(T'') > weight(T), so T is a unique minimal spanning tree. $\square$

Question 5 continues on the next page...

(b) (4 points) Give an example of a connected weighted graph G so that all the edges of G have distinct weights and G has at least two distinct spanning trees that have the same total weight, i.e. the sums of the weights of the edges in these two distinct trees agree, *or* prove that no such weighted graph exists.

G

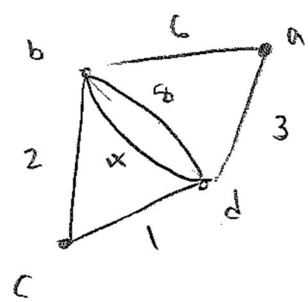


~~No such weighted graph exists.~~

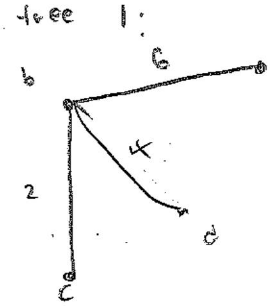
~~In the previous example, we showed that in a connected graph G with all edges having distinct weights will be a unique minimal spanning tree.~~

~~tradiction, let us say that a graph like ·) has two identical sums for different trees. These spanning~~

G

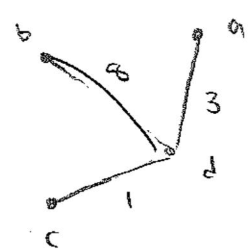


tree 1:



sum = 12

tree 2:



sum = 12

6. (a) (3 points) Show that for G a connected simple planar graph containing a cycle if G has E edges and F faces, then $2E \geq 3F$.

If G is a connected simple planar graph, then $2E \geq 3F$. This is because at least 3 edges are needed to form a cycle, which makes a face, and each edge is a boundary for 2 faces (hence $2E$), so they are counted twice.

(The $\geq$ sign accounts for stray edges that are not part of any cycle, ie: )

(b) (3 points) Show that for G a connected simple planar graph containing a cycle if G has E edges and V vertices, then $E \leq 3V - 6$.

As shown above, $2E \geq 3F$

According to Euler's theorem, if G is a planar graph, then $V - E + F = 2$,

We can use these two expressions to get $E \leq 3V - 6$.

$2E \geq 3F$

$V - E + F = 2$

$F = 2 + E - V$

$2E \geq 3(2 + E - V)$

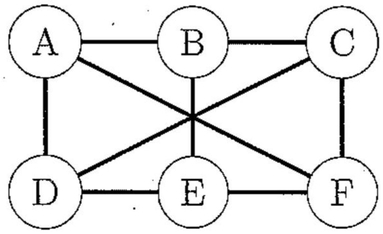
$2E \geq 6 + 3E - 3V$

$3V - 6 \geq E$

$$\boxed{E \leq 3V - 6}$$

Question 6 continues on the next page...

(c) (4 points) Is the following graph planar? If it is give a planar drawing of it. If not, prove that it is not planar.



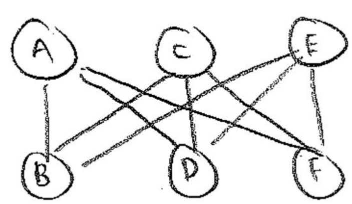
not planar; it is $K_{3,3}$

We can split the vertices into two disjoint sets, V_1 and V_2

$V_1 = \{A, C, E\}$ and $V_2 = \{B, D, F\}$.

A, C, and E do not have edges connecting them, and B, D, F do not either, but A, C, E are connected by edges to all of B, D, and F, so it is a bipartite graph.

The above graph can be redrawn as such:



this is $K_{3,3}$

therefore, not planar.

7. (a) (5 points) Show that for all $n \geq 1$, $7^n - 1$ is divisible by 6.

base case: $n=1$ $\quad 7^1 - 1 = 6$ which is divisible by 6

base case is true ✓

inductive step: we know $7^n - 1$ is divisible by 6 for $n \geq 1$

want to show $7^{n+1} - 1$ is divisible by 6.

$$7^{n+1} - 1 = 7^{n+1} - 7 + 6$$
$$= \underbrace{7(7^n - 1)}_{\text{divisible by } 6} + \underbrace{6}_{6 \text{ is divisible by } 6}$$

Any multiple of an integer that is divisible by 6 is also divisible by 6. Additionally, the sum of two integers div. by 6 is also div. by 6. Therefore, $7(7^n - 1) + 6$ is divisible by 6. □

(b) (5 points) Show that there is a number of the form $\sum_{i=0}^{n} 10^i$ (i.e. a number consisting only of 1s) that is divisible by 7.

$n = 1$

$\sum_{i=0}^{1} 10^i = 1 + 10 = 11$

5714

pigeonhole? .

$$\begin{array}{r} 15873 \\ 7 \overline{)111111} \\ -7 \\ \hline 41 \\ -35 \\ \hline 61 \\ -56 \\ \hline 51 \\ -49 \\ \hline 21 \\ 21 \\ \hline 0 \end{array}$$

4, 6, 5, 2,

There are 7 possible remainders when dividing by 7: 0, ..., 6.

Because of this, there will eventually be a number of the form $\sum_{i=0}^{n} 10^i$ that is divisible by 7, because eventually, there will be a remainder of 0 (pigeonhole principle).

Ex: $n = 5$ $\quad \sum_{i=0}^{5} 10^i = 111111$ (see above left) is divisible by 7.

8. A *balanced binary tree* is a binary tree where for each vertex the heights of the left and right subtrees of that vertex differ by at most one. Let v_n denote the minimum number of vertices in a balanced binary tree of height n.

(a) (4 points) Show that v_n satisfies for $n \geq 2$ the recurrence $v_n = v_{n-1} + v_{n-2}$. $+ 1$

If V_n is the minimum number of vertices in a balanced binary tree of height n, and $n \geq 2$, then the root vertex of V_n has a left and right subtree that differ by at most one. One subtree has a V_{n-1} minimum # of vertices with height n-1, and the other has V_{n-2} minimum number of vertices w/ height n-2 (since they differ by at most 1).

Thus, the recurrence relation shows that the minimum number of vertices in a balanced binary tree of height n is the sum of the minimum # of vertices in the left and right subtree of the root vertex's children, plus the root vertex. $V_n = V_{n-1} + V_{n-2}$

$V_0 = 1$ ← the root vertex $\quad V_1 = 2 \quad V_2 = 1+2+1 = 4$

(b) (3 points) Show that for $n \geq 0$, $v_n = F_{\cancel{n}+2}$, where F_k is the k^{th} Fibonacci number. $F_{n+3} - 1$

$\therefore V_n = F_{n+3} - 1$

$F_0 = 0 \quad F_1 = 1 \quad F_2 = 1 \quad F_3 = 2$

$F_{n+3} = V_n + 1$

$V_0 = F_3 - 1 = 2 - 1 = 1$

Fibonacci sequence follows the recurrence relation

$V_1 = F_4 - 1 = 3 - 1 = 2$

$V_2 = F_5 - 1 = 5 - 1 = 4$

$F_{n+3} = F_{n+2} + F_{n+1}$

$V_n + 1 = V_{n-1} + 1 + V_{n-2} + 1$

V_n follows the same recurrence relation

$V_n + 1 = V_{n-1} + V_{n-2} + 2$

$V_n = V_{n-1} + V_{n-2} + 1$ ← this is shown in the above example to be true

□

Question 8 continues on the next page...

(c) (3 points) Show that $v_n = \Theta(\phi^{n+2})$, where $\phi = \frac{1+\sqrt{5}}{2}$.

$$V_n = V_{n-1} + V_{n-2} + 1 \geq V_{n-1} + V_{n-2}$$

$$V_n = V_{n-1} + V_{n-2} + 1$$

solve w/ constant and w/o constant

let $V_n = X^2$

$$x^2 = x + 1$$

$$x^2 - x - 1 = 0$$

$$\frac{1 \pm \sqrt{1+4}}{2}$$

$$x = \frac{1 \pm \sqrt{5}}{2}$$

$$V_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$V_0 = 1$

$V_1 = 2$

$$1 = A + B$$

$$2 = \left(\frac{1+\sqrt{5}}{2}\right)A + \left(\frac{1-\sqrt{5}}{2}\right)B$$

$$\left(\frac{1+\sqrt{5}}{2}\right) = \left(\frac{1+\sqrt{5}}{2}\right)A + \left(\frac{1+\sqrt{5}}{2}\right)B$$

$$\left(\frac{1+\sqrt{5}}{2} = \frac{4}{2}\right) = \left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right)B$$

$$\frac{1}{\sqrt{5}} \cdot \frac{\sqrt{5}-3}{2} = \frac{8\sqrt{5}}{2} B$$

$$\frac{-3+\sqrt{5}}{2\sqrt{5}} \qquad \frac{-3\sqrt{5}+5}{10}$$

$$A = \frac{1}{\sqrt{5}} \qquad B = \frac{-1}{\sqrt{5}}$$

$$V_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{-1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n \nearrow \quad n+2 > n$$

$$\therefore \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{-1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n \leq \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} = O(\phi^{n+2})$$

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n \geq \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} = \Omega(\phi^{n+2}) \quad \text{so} \quad V_n = \Theta(\phi^{n+2})$$

9. (a) (4 points) Show that $\sum_{i=0}^{n} 2^i \binom{n}{i} = 3^n$.

by the binomial theorem:

$$(1+2)^n = \sum_{i=0}^{n} \binom{n}{i} \cancel{1}^{n-i} 2^i$$

So $$3^n = \sum_{i=0}^{n} \binom{n}{i} 2^i$$

✓

In other words, 3^n is when you have two subsets of X, A and B, where $A \subseteq B \subseteq X$.

Each of the n elements has 3 options. It can be in

1) Both A and B
2) In B but not A
3) In neither A nor B

$\sum_{i=0}^{n} 2^i \binom{n}{i}$ Shows the options after making a subset B of n, 2^i (the element is there or not there), adding up the number of elements in the subset A multiplied by the possible subsets 2^i.

(b) (6 points) Show that $\binom{n+m}{r} = \sum_{i=0}^{r} \binom{n}{i} \binom{m}{r-i}$.

$$\binom{n+m}{r} = \sum_{i=0}^{r} \binom{n}{i} \binom{m}{r-i}$$

$$\binom{n+m}{r} = \binom{n+m-1}{r} + \binom{n+m-1}{r-1}$$

$$\sum_{i=0}^{r} \frac{n!}{i!(n-i)!} + \frac{m!}{(r-i)!(m-r+i)!}$$

$$\frac{(n+m)!}{r!(n+m-r)!}$$

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